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Multivariable orthogonal polynomials  
and quantum Grassmannians

J.V. Stokman

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**Editorial Preface**

This CWI Tract contains the PhD dissertation of Dr. Jasper V. Stokman, as it has been defended at the University of Amsterdam on June 11, 1998. The research of this thesis has been performed within the framework of the Dutch Research School 'Thomas Stieltjes Institute for Mathematics'. The Stieltjes Institute has awarded Dr. Stokman's thesis with a prize, as being the best 1998 thesis written in this Research School. We congratulate Dr. Stokman with this prize, and appreciate that we have obtained his permission to include his thesis into the CWI Tract Series.

The editors of CWI Tracts

## Foreword

The intimate relation between representation theory and the theory of special functions is a continuing source of new and beautiful results in both fields of mathematics. An example of the interaction between representation theory and special functions is the development of a general theory on hypergeometric functions associated with root systems by Heckman and Opdam, which was motivated by their interpretation (for very special parameter values) as zonal spherical functions on Riemannian symmetric spaces. Another more recent example is the representation theoretic interpretation of Macdonald polynomials associated with root systems. The Macdonald polynomials were related to representation theory of affine Hecke algebras by Cherednik and to harmonic analysis on certain quantizations of homogeneous spaces by Noumi and Sugitani. The last example is an illustration of the remarkable fact that the “ $q$ ” in  $q$ -special function theory is essentially the same  $q$  as in quantum groups, if the “right”  $q$ -deformations are chosen.

One of the interesting new aspects of the harmonic analysis on quantized homogeneous spaces is the implicit role played by the Poisson structure on the underlying spaces. The Poisson structure is built in the quantization, so inequivalent Poisson structures give rise to different quantizations. Hence the harmonic analysis on quantizations of homogeneous spaces depends on the choice of Poisson structures on the underlying spaces. A nice illustration of this phenomenon is the interpretation of several families of orthogonal polynomials on different quantizations of the 2-sphere by Koornwinder, Mimachi, Noumi and others.

Poisson structures also play an important role in the representation theory of the quantized homogeneous spaces themselves. The origin of this observation lies in the orbit method of Kostant, Kirillov and Souriau in which irreducible unitary representations of Lie groups are related to coadjoint orbits of the corresponding Lie algebras. Coadjoint orbits are symplectic submanifolds for the Kostant-Kirillov Poisson bracket. The quantum orbit method deals with the problem of relating representation theory of arbitrary quantized Poisson algebras to the Poisson geometry of the underlying spaces. A beautiful example of the quantum orbit method is Soibelman’s classification of the irreducible  $*$ -representations of the quantized function algebras on compact simple Lie groups. Many properties of the irreducible  $*$ -representations were shown to be closely related to geometric properties of the underlying Poisson-Lie groups.

In the present CWI Tract the above mentioned ideas are developed further in several different directions. It is entirely based on my dissertation which I have completed in June, 1998 at the KdV institute for Mathematics, University of Amsterdam under the supervision of Prof. Tom H. Koornwinder. The main text of my dissertation is reproduced here without major changes; I have corrected some misprints, and updated the references.

*Acknowledgments:* I thank my former thesis-advisor Tom H. Koornwinder for his advise and guidance during the four years that I have worked on my dissertation. I am financially supported by a fellowship from the Royal Netherlands Academy of Arts and Sciences (KNAW).



## CHAPTER 1

# General introduction

### 1.1. Introduction

The present Chapter contains a general introduction to the different topics of the Tract. I aim to clarify some of the main ideas and techniques by considering certain simplified examples in detail.

In Chapter 2 and 3 of the Tract certain families of multivariable orthogonal polynomials are studied. A main tool in the analysis is the development of a residue calculus for a specific multidimensional contour integral. In Section 1.2 we illustrate some of these techniques for certain  $q$ -analogues of Euler's beta integral.

The remaining chapters of the Tract rely on representation theoretic methods, which are mostly of algebraic nature. In Chapter 5 and Chapter 6 intensive use is made of generalizations of Plücker coordinates. As an illustration, I give in Section 1.3 the construction of Plücker coordinates on complex Grassmannians. I only use here some well-known notions from algebraic geometry and algebraic groups, which can for instance be found in [112].

In Section 1.4 I give a detailed description of the contents of the Tract while referring to the simplified examples given in Section 1.2 and Section 1.3. The notations and conventions which are used throughout the Tract, are listed in Section 1.5.

### 1.2. $q$ -Analogues of Euler's beta integral

A well-known identity in special function theory is Euler's beta integral,

$$(1.2.1) \quad \int_0^1 z^\alpha (1-z)^\beta dz = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, \quad (\operatorname{Re}(\alpha), \operatorname{Re}(\beta) > -1)$$

where  $\Gamma(z)$  is the Gamma-function

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad (\operatorname{Re}(z) > 0).$$

In this section a discrete and a continuous  $q$ -deformation of (1.2.1) is considered, where  $q$  is a fixed real number in the open interval  $(0, 1)$ . An identity (depending on  $q$ ) is called a  $q$ -deformation or a  $q$ -analogue of Euler's beta integral when Euler's beta integral (1.2.1) can be recovered by taking the limit  $q \uparrow 1$  in a suitable manner. The  $q$ -analogue is called discrete (respectively continuous) if it is an evaluation of a possibly infinite sum (respectively contour integral).

The theory of  $q$ -special functions has its origin in the work of Euler, Gauss, Jacobi and in particular Heine, who derived several fundamental properties of a  $q$ -analogue of Gauss's hypergeometric series. Nowadays  $q$ -deformations have been found for many identities in classical special function theory. In particular, several  $q$ -analogues of Euler's beta integral (1.2.1) have been found. The discrete  $q$ -analogue of Euler's beta integral which are considered in this section can be most conveniently expressed in terms of Jackson's  $q$ -integral,

$$(1.2.2) \quad \int_c^d f(z) d_q z := \int_0^d f(z) d_q z - \int_0^c f(z) d_q z,$$

$$\int_0^c f(z) d_q z := (1-q) \sum_{i=0}^{\infty} f(cq^i) cq^i,$$

which are considered here for functions  $f$  such that both sums are absolutely convergent. Observe that the  $q$ -integral of a continuous function  $f$  over the interval  $[c, d]$  tends to the usual Riemann integral of  $f$  over  $[c, d]$  when  $q \uparrow 1$ . The discrete  $q$ -beta integral is given by

$$(1.2.3) \quad \int_0^1 \frac{(qz; q)_{\infty}}{(q^{\beta+1}z; q)_{\infty}} z^{\alpha} d_q z = \frac{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}{\Gamma_q(2+\alpha+\beta)}$$

( $\alpha, \beta > -1$ ), where  $(a; q)_k$  ( $k \in \mathbb{Z}_+ \cup \{\infty\}$ ) is the  $q$ -shifted factorial,

$$(a; q)_k := \prod_{i=0}^{k-1} (1 - aq^i), \quad (a; q)_{\infty} := \lim_{k \rightarrow \infty} (a; q)_k$$

and  $\Gamma_q(z)$  is the  $q$ -Gamma function

$$(1.2.4) \quad \Gamma_q(z) := \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} (1-q)^{1-z}, \quad z \notin -\mathbb{Z}_+,$$

where  $\mathbb{Z}_+$  is the set of positive integers. Observe that  $z \in -\mathbb{Z}_+$  should be excluded in the definition of  $\Gamma_q(z)$  since  $(q^z; q)_{\infty} = 0$  for  $z \in -\mathbb{Z}_+$ . The discrete  $q$ -beta integral (1.2.3) tends formally to Euler's beta integral (1.2.1) in the limit  $q \uparrow 1$  since the  $q$ -Gamma function  $\Gamma_q(z)$  tends to the Gamma function  $\Gamma(z)$  when  $q \uparrow 1$ , and

$$\lim_{q \uparrow 1} \frac{(qz; q)_{\infty}}{(q^{\beta+1}z; q)_{\infty}} = (1-z)^{\beta}.$$

A very general continuous  $q$ -analogue of the beta integral which depends, besides  $q$ , on four additional parameters  $\underline{t} = (t_0, t_1, t_2, t_3)$ , is the Askey-Wilson integral [7]

$$(1.2.5) \quad \frac{1}{2\pi i} \int_{z \in T} \frac{(z^2, z^{-2}; q)_{\infty}}{(t_0 z, t_0 z^{-1}, t_1 z, t_1 z^{-1}, t_2 z, t_2 z^{-1}, t_3 z, t_3 z^{-1}; q)_{\infty}} \frac{dz}{z}$$

$$= \frac{2(t_0 t_1 t_2 t_3; q)_{\infty}}{(q, t_0 t_1, t_0 t_2, t_0 t_3, t_1 t_2, t_1 t_3, t_2 t_3; q)_{\infty}}$$

where the parameters  $t_i$  are generic complex with moduli  $< 1$  and with  $T$  the unit circle in the complex plane. Here the shorthand notation  $(a_1, \dots, a_r; q)_k := (a_1; q)_k \dots (a_r; q)_k$  for products of  $q$ -shifted factorials is used. Although this is not clear at first sight, the beta integral (1.2.1) is a limit case of the Askey-Wilson integral (1.2.5). This can be shown by making the change of variables  $x = z + z^{-1}$ , substituting  $t_0 = q^{\alpha + \frac{1}{2}}$ ,  $t_1 = -q^{\beta + \frac{1}{2}}$ ,  $t_2 = q^{\frac{1}{2}}$  and  $t_3 = -q^{\frac{1}{2}}$  and taking the limit  $q \uparrow 1$  in (1.2.5). Then, we obtain

$$\int_{-1}^1 (1-x)^\alpha (1+x)^\beta dx = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

which is equivalent with (1.2.1) (cf. [30, Section 6.1]).

It turns out that the discrete  $q$ -beta integral (1.2.3) can be considered as a limit case of the Askey-Wilson integral (1.2.5). This is illustrated here for special parameter values. The  $q$ -beta integral (1.2.3) for  $\alpha = -1/2$  and  $\beta = \infty$  (i.e.  $q^\beta = 0$ ) is equivalent to the summation formula

$$(1.2.6) \quad \sum_{k=0}^{\infty} \frac{q^{k/2}}{(q; q)_k} = \frac{1}{(q^{\frac{1}{2}}; q)_{\infty}}$$

which is a special case of the  $q$ -binomial theorem

$$\sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k = \frac{(az; q)_{\infty}}{(z; q)_{\infty}}, \quad (|z| < 1).$$

The summation formula (1.2.6) can be derived from the Askey-Wilson integral (1.2.5) by analytic continuation, residue computation, and a limit transition as follows. Specializing the parameters by

$$(t_0, t_1, t_2, t_3) = (\varepsilon^{-1} q^{\frac{1}{2}}, -1, 0, -q^{\frac{1}{2}})$$

in (1.2.5) and using that  $(z^2; q)_{\infty} = (z, -z, q^{\frac{1}{2}}z, -q^{\frac{1}{2}}z; q)_{\infty}$ , we obtain

$$(1.2.7) \quad \frac{1}{2\pi i} \int_{z \in T} w_c(z; \varepsilon) dz = \frac{2}{(q, -\varepsilon^{-1} q^{\frac{1}{2}}, -\varepsilon^{-1} q, q^{\frac{1}{2}}; q)_{\infty}}$$

for  $\varepsilon > q^{-\frac{1}{2}}$  with weight function  $w_c(z; \varepsilon)$  given by

$$w_c(z; \varepsilon) := \frac{(z, q^{\frac{1}{2}}z, z^{-1}, q^{\frac{1}{2}}z^{-1}; q)_{\infty}}{z(\varepsilon^{-1} q^{\frac{1}{2}}z, \varepsilon^{-1} q^{\frac{1}{2}}z^{-1}; q)_{\infty}}.$$

Now  $w_c(\cdot; \varepsilon)$  for generic  $\varepsilon > 0$  has two infinite sequences of simple poles, namely the converging sequence of poles  $\{\varepsilon^{-1} q^{\frac{1}{2}+k}\}_{k \in \mathbb{Z}_+}$  respectively the diverging sequence of poles  $\{\varepsilon q^{-\frac{1}{2}-k}\}_{k \in \mathbb{Z}_+}$ . By analytic continuation, (1.2.7) is valid for arbitrary  $\varepsilon > 0$  if the integration contour  $T$  is replaced by a rectifiable Jordan curve  $T_\varepsilon$  for which the converging sequence of poles is contained in its interior and the diverging sequence of poles is contained in its exterior. Let  $w_d(k; \varepsilon)$  ( $k \in \mathbb{Z}_+$ ) be the residue of  $w_c(z; \varepsilon)$  at  $z = \varepsilon^{-1} q^{\frac{1}{2}+k}$ .

By an easy computation, the residues  $w_d(k; \varepsilon)$  for generic  $\varepsilon$  are computed explicitly as

$$w_d(k; \varepsilon) = \left( \frac{(\varepsilon^{-2}q; q)_k}{(\varepsilon^{-1}q^{\frac{1}{2}}, \varepsilon^{-1}q; q)_k} \right) \frac{1}{(-\varepsilon^{-1}q^{\frac{1}{2}}, -\varepsilon^{-1}q; q)_\infty (q; q)_\infty (q^{-k}; q)_k}$$

for  $k \in \mathbb{Z}_+$ . Observe that the residue of  $w_c(z; \varepsilon)$  at  $z = \varepsilon q^{-\frac{1}{2}-k}$  equals  $-w_d(k; \varepsilon)$ . For generic  $\varepsilon > 0$ , the contour  $T_\varepsilon$  in  $\int_{T_\varepsilon} w_c(z; \varepsilon) dz$  can now be pulled back to the unit circle  $T$  while picking up residues. Then, it follows for generic  $\varepsilon > 0$  by Cauchy's theorem that

$$(1.2.8) \quad \frac{1}{2\pi i} \int_{z \in T} w_c(z; \varepsilon) dz + 2 \sum_{k \in \mathbb{Z}_+ : \varepsilon < q^{\frac{1}{2}+k}} w_d(k; \varepsilon) = \frac{2}{(q, -\varepsilon^{-1}q^{\frac{1}{2}}, -\varepsilon^{-1}q, q^{\frac{1}{2}}; q)_\infty}.$$

Observe that  $w_d(k; \varepsilon)$  and the right hand side of (1.2.8) have  $(-\varepsilon^{-1}q^{\frac{1}{2}}, -\varepsilon^{-1}q; q)_\infty$  as a common factor in their denominators. Now multiply the left and right hand side of (1.2.8) with this common factor and take the limit  $\varepsilon \downarrow 0$ . By Lebesgue's dominated convergence theorem, the continuous part in the left hand side of (1.2.8) disappears in this limit, while the finite sum in the left hand side of (1.2.8) tends to the infinite sum

$$2 \sum_{k \in \mathbb{Z}_+} \frac{q^{k/2}}{(q; q)_k (q; q)_\infty}.$$

It follows that the remaining summation formula is equivalent to (1.2.6). This type of computation can be done for general  $\alpha, \beta > -1$  (see [122] and Chapter 3), which then shows that the discrete  $q$ -beta integral (1.2.3) is a limit case of the continuous  $q$ -beta integral (1.2.5).

### 1.3. Plücker coordinates on the Grassmann manifold

Let  $l$  and  $n$  be non-zero positive integers such that  $l \leq [n/2]$  and let  $Y_{l,n}$  be the set of  $l$  dimensional subspaces of  $V := \mathbb{C}^n$ . In the first part of this section we associate to  $Y_{l,n}$  an algebra of functions, the so-called homogeneous coordinate ring of  $Y_{l,n}$ . In order to define the homogeneous coordinate ring of  $Y_{l,n}$ , we first need to give a different description of  $Y_{l,n}$ .

The tensor algebra  $T(V)$  of the vector space  $V$  is the linear space

$$T(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

with multiplication defined by  $t.t' := t \otimes t'$  ( $t, t' \in T(V)$ ). The direct sum decomposition  $T(V) = \bigoplus_{m \in \mathbb{Z}_+} V^{\otimes m}$  is a grading with respect to this multiplication.

The exterior algebra  $\Lambda(V)$  of the vector space  $V$  is by definition the quotient of  $T(V)$  with the two-sided ideal  $\mathcal{J}$  generated by  $v \otimes v$  ( $v \in V$ ). The product in  $\Lambda(V)$  is called the wedge product, and is denoted by  $\wedge$ .

Let  $\{v_i\}_{i=1}^n$  be the canonical basis of  $V$  and let  $P_m(n)$  be the collection of subsets of  $\{1, \dots, n\}$  of cardinality  $m$ . The elements

$$v_J := v_{j_1} \wedge v_{j_2} \wedge \dots \wedge v_{j_m} \in \Lambda(V)$$

with  $J = \{j_1 < \dots < j_m\} \in P_m(n)$  and  $0 \leq m \leq n$  form a linear basis for  $\Lambda(V)$ . Here we have used the convention that  $v_\emptyset := 1$ . The direct sum decomposition

$$\Lambda(V) = \bigoplus_{m=0}^n \Lambda^m(V),$$

with  $\Lambda^m(V)$  the span of the basis elements  $v_J$  ( $J \in P_m(n)$ ) is a grading of  $\Lambda(V)$  with respect to the wedge product.

Let  $X := \mathbb{P}(\Lambda^l(V))$  be the projective space associated with  $\Lambda^l(V)$ , i.e.  $X$  is the collection of one dimensional subspaces of  $\Lambda^l(V)$ . The natural projection  $\pi : \Lambda^l(V) \setminus \{0\} \rightarrow X$  sends  $0 \neq w \in \Lambda^l(V)$  to the one dimensional subspace  $[w]$  containing  $w$ . Now  $Y_{l,n}$  can be embedded in  $X$  via the so-called Plücker embedding. The Plücker embedding is defined by

$$Y_{l,n} \rightarrow X, \quad W = \text{span}\{w_1, \dots, w_l\} \mapsto [w_1 \wedge \dots \wedge w_l].$$

It is easily seen that the Plücker embedding is well-defined and injective.

It turns out that there is a nice description of the image of  $Y_{l,n}$  under the Plücker embedding. To give this description of the image, we first need to introduce a group action on the projective space  $X$ .

Let  $G := GL(n, \mathbb{C})$  be the group of  $n$  by  $n$  invertible matrices over  $\mathbb{C}$ . It is well-known that  $G$  has the structure of an irreducible affine variety, i.e.  $G$  is a connected linear algebraic group. The group  $G$  acts on  $V$  by the usual matrix multiplication

$$(1.3.1) \quad g.v_j := \sum_i \xi_j^i(g) v_i, \quad g \in G,$$

where  $\xi_j^i$  is the coordinate function on  $G$  defined by  $\xi_j^i(g) = g_{ij}$  if  $g = (g_{ij})_{i,j} \in G$ .

The left  $G$ -action (1.3.1) on  $V$  extends to a  $G$ -action on  $\Lambda(V)$  as follows. The tensor algebra  $T(V)$  has a unique left  $G$ -module algebra structure such that its action on  $V$  coincides with the action (1.3.1). In particular, this means that  $g.(v \otimes w) = g.v \otimes g.w$  for  $g \in G$  and  $v, w \in T(V)$ . It follows that the two-sided ideal  $\mathcal{J}$  is stable under the  $G$ -action, i.e.  $g.\mathcal{J} \subset \mathcal{J}$  for all  $g \in G$ . Consequently, the exterior algebra  $\Lambda(V)$  inherits a  $G$ -module algebra structure from the  $G$ -action on the tensor algebra  $T(V)$ .

Explicitly, the action of  $G$  on the linear basis  $\{v_I \mid I \in P_m(n)\}$  of  $\Lambda^m(V)$  is given by

$$(1.3.2) \quad g.v_J = \sum_{I \in P_m(n)} \xi_J^I(g) v_I, \quad J \in P_m(n),$$

where  $\xi_J^I(g)$  is the determinant of the  $m$  by  $m$  submatrix of  $g$  obtained from  $g$  by deleting all rows with row index  $I^c := \{1, \dots, n\} \setminus I$  and by deleting all columns with column

index  $J^c$ . Formula (1.3.2) for the action on  $\Lambda^m(V)$  is a direct consequence of the well-known expansion formula for determinants

$$(1.3.3) \quad \xi_J^I = \sum_{\sigma \in \mathfrak{S}_m} (-1)^{l(\sigma)} \xi_{j_1}^{i_{\sigma(1)}} \xi_{j_2}^{i_{\sigma(2)}} \cdots \xi_{j_m}^{i_{\sigma(m)}},$$

where  $I = \{i_1 < \dots < i_m\}$ ,  $J = \{j_1 < \dots < j_m\} \in P_m(n)$ ,  $\mathfrak{S}_m$  is the permutation group of  $\{1, \dots, m\}$  and

$$l(\sigma) := \#\{1 \leq i < j \leq m \mid \sigma(i) > \sigma(j)\}$$

is the so-called length function on  $\mathfrak{S}_m$ . In particular, it follows from (1.3.2) that the graded pieces  $\Lambda^m(V)$  are invariant subspaces for the  $G$ -action. Observe that

$$(1.3.4) \quad g.[w] := [g.w], \quad (g \in G, w \in \Lambda^m(V)),$$

is a well-defined left  $G$ -action on  $X$ . Set

$$x := v_{\{n-l+1, \dots, n\}} \in \Lambda^l(V),$$

then it follows by elementary linear algebra that the  $G$ -orbit  $G.[x] := \{g.[x] \mid g \in G\} \subset X$  is exactly the image of  $Y_{l,n}$  under the Plücker embedding. From now on, we identify  $Y_{l,n}$  with its image under the Plücker embedding, i.e. we identify  $Y_{l,n}$  with the  $G$ -orbit  $G.[x]$ .

Observe that via the map  $G \rightarrow G.[x]$ ,  $g \mapsto g.[x]$ , the  $G$ -orbit  $Y_{l,n} = G.[x]$  can be identified with the coset space  $G/G_x := \{gG_x \mid g \in G\}$ , where

$$G_x := \{g \in G \mid g.[x] = [x]\}$$

is the so-called isotropy subgroup of  $[x]$ . It is not difficult to verify that

$$(1.3.5) \quad G_x = \{g = (g_{ij})_{i,j} \in G \mid g_{ij} = 0 \text{ for } 1 \leq i \leq n-l, n-l+1 \leq j \leq n\}.$$

In particular, the lower triangular matrices are contained in the isotropy subgroup  $G_x$ . This fact implies that  $G_x$  is a so-called parabolic subgroup. A parabolic subgroup  $P$  of  $G$  has the important property that the corresponding quotient  $G/P$  is a projective  $G$ -variety, i.e. the quotient  $G/P$  can be realized as the zero set of a finite collection of homogeneous polynomials. So  $Y_{l,n} = G.[x]$  becomes a  $(n-l)l$  dimensional irreducible projective variety, the so-called (complex) *projective Grassmann manifold*.

The homogeneous coordinate ring  $A$  of  $Y_{l,n} \subset \mathbb{P}(\Lambda^l(V))$  is defined as

$$A := \mathbb{C}[\Lambda^l(V)]/\mathfrak{I},$$

where  $\mathbb{C}[\Lambda^l(V)]$  is the coordinate ring of  $\Lambda^l(V)$  and  $\mathfrak{I}$  is the ideal generated by the homogeneous polynomials  $f \in \mathbb{C}[\Lambda^l(V)]$  which vanish on  $Y_{l,n}$ . Equivalently,  $A$  is the coordinate ring of the affine cone  $Y_{l,n}^a := \pi^{-1}(Y_{l,n}) \cup \{0\} \subset \Lambda^l(V)$  over  $Y_{l,n}$ , where  $\pi : \Lambda^l(V) \setminus \{0\} \rightarrow X$  is the natural projection. The algebra  $A$  can be realized as a subalgebra of the coordinate ring  $\mathbb{C}[G]$  of  $G$  as follows. Consider the morphism  $\phi : G \rightarrow Y_{l,n}^a$  defined by  $\phi(g) := g.x$ . The dual mapping  $\phi^*$  is by definition the algebra homomorphism

$$\phi^* : A \rightarrow \mathbb{C}[G], \quad \phi^*(f) := f \circ \phi.$$

The dual map  $\phi^*$  is injective since  $\overline{\phi(G)} = Y_{l,n}^a$ , where  $\overline{\phi(G)}$  is the closure of  $\phi(G)$  in  $Y_{l,n}^a$ . Hence  $\phi^*$  embeds  $A$  into the coordinate ring  $\mathbb{C}[G]$  of  $G$ . We can give now an explicit set of algebraic generators for the image of  $\phi^*$  by computing the image under  $\phi^*$  of the coordinate functions on  $\Lambda^l(V)$ . We take here the coordinate functions  $\eta_I \in \mathbb{C}[\Lambda^l(V)]$  ( $I \in P_l(n)$ ) with respect to the linear basis  $\{v_I \mid I \in P_l(n)\}$  of  $\Lambda^l(V)$ , i.e.  $\eta_I : \Lambda^l(V) \rightarrow \mathbb{C}$  is the linear mapping which maps the basis elements  $v_J$  ( $J \neq I$ ) to 0 and which maps  $v_I$  to 1. Then,

$$(\phi^*(\eta_I))(g) = \eta_I(g.x) = t_I(g), \quad (I \in P_l(n)),$$

where the  $t_I$  ( $I \in P_l(n)$ ) are the so-called *Plücker coordinates*

$$(1.3.6) \quad t_I := \xi_{\{n-l+1, \dots, n\}}^I, \quad (I \in P_l(n)).$$

In other words, the algebra  $A$  may be identified with the subalgebra  $\mathbb{C}[t_I \mid I \in P_l(n)]$  of  $\mathbb{C}[G]$  generated by the Plücker coordinates  $t_I$  ( $I \in P_l(n)$ ).

A set of homogeneous algebraic relations between the Plücker coordinates  $t_I$  are called a set of defining relations if its pre-image under  $\phi^*$  generates  $\mathcal{I}$  as an ideal. We describe here one well-known set of defining relations, namely the set of Plücker relations. To give these quadratic relations it is convenient to define for arbitrary  $i_1, \dots, i_l$  between 1 and  $n$ ,

$$t_{i_1, \dots, i_l} := \sum_{\sigma \in \mathfrak{S}_l} (-1)^{l(\sigma)} \xi_{n-l+1}^{i_{\sigma(1)}} \xi_{n-l+2}^{i_{\sigma(2)}} \dots \xi_n^{i_{\sigma(l)}}.$$

Observe that  $t_{i_1, \dots, i_l}$  is anti-symmetric in the  $l$  indices  $i_1, \dots, i_l$  and that its definition coincides with the definition (1.3.6) of the Plücker coordinate  $t_{\{i_1, \dots, i_l\}}$  if  $i_1 < \dots < i_l$ . In particular,  $t_{i_1, \dots, i_l} = 0$  when two indices  $i_p, i_r$  ( $p \neq r$ ) coincide. The Plücker relations are now given by the quadratic relations

$$(1.3.7) \quad \sum_{i=1}^{l+1} (-1)^i t_{k_1, \dots, \hat{k}_i, \dots, k_{l+1}} t_{k_i, j_1, \dots, j_{l-1}} = 0$$

for subsets  $K = \{k_1 < \dots < k_{l+1}\} \in P_{l+1}(n)$  and  $J = \{j_1 < \dots < j_{l-1}\} \in P_{l-1}(n)$ , where  $\hat{k}$  means that the element  $k$  should be omitted. There is a rich combinatorial structure related to the defining relations of the Grassmann manifold, for which we refer to Towber [128].

The so-called Levi subgroup  $L$  corresponding to the maximal parabolic subgroup  $G_x$  consists of the  $n$  by  $n$  invertible matrices of the form

$$(1.3.8) \quad \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \quad B \in GL(n-l, \mathbb{C}), \quad C \in GL(l, \mathbb{C}).$$

The algebra of right  $L$ -invariant polynomial functions on  $G$ ,

$$(1.3.9) \quad \mathbb{C}[G/L] := \{f \in \mathbb{C}[G] \mid f(gp) = f(g), \quad \forall g \in G, \forall p \in L\},$$

turns out to be closely related to the homogeneous coordinate ring  $A$  of the projective Grassmann manifold  $Y_{l,n}$ . Since the precise connection between  $A$  and  $\mathbb{C}[G/L]$  in a

more general setting plays a fundamental role in the last two chapters of the Tract, we explain it here in more detail for the complex Grassmannian.

For the precise connection between  $A$  and  $\mathbb{C}[G/L]$  we have to consider a specific  $G$ -orbit within  $\Lambda^l(V) \otimes \Lambda^l(V^*)$ , where  $V^*$  is the dual module of  $V$ . First, we define the dual module of  $V$  and we give the dual construction of the Grassmannian manifold  $Y_{l,n}$  by replacing the role of the  $G$ -module  $V$  by its dual. Detailed discussions are omitted here since the constructions and techniques are the same as before.

The  $G$ -action on the linear dual  $V^* := \{f : V \rightarrow \mathbb{C} \mid f \text{ linear}\}$  of  $V$  is given by

$$(g.f)(v) := f(g^{-1}.v), \quad (f \in V^*, v \in V, g \in G).$$

Let  $\{v_i^*\}_i$  be the dual basis with respect to  $\{v_i\}_i$ . A linear basis of the  $m$ th graded part of the exterior algebra  $\Lambda(V^*)$  is given by the elements

$$v_I^* := v_{i_1}^* \wedge v_{i_2}^* \wedge \dots \wedge v_{i_m}^*,$$

where  $I = \{i_1 < \dots < i_m\} \in P_m(n)$ . Set

$$\bar{\xi}_J^I(g) := \xi_J^I(g^{-1}) \quad (g \in G \text{ and } I, J \in P_m(n)),$$

then the  $\bar{\xi}_J^I$  occur as matrix coefficients of the induced  $G$ -action on  $\Lambda^m(V^*)$ ,

$$g.v_J^* = \sum_{I \in P_m(n)} \bar{\xi}_J^I(g) v_I^*, \quad (J \in P_m(n)).$$

Now consider the action of  $G$  on the projective space  $X^* := \mathbb{P}(\Lambda^m(V^*))$  and let  $Y_{l,n}^*$  be the  $G$ -orbit of  $[x^*] \in X^*$ , where

$$x^* := v_{\{n-l+1, \dots, n\}}^* \in \Lambda^l(V^*).$$

Then  $Y_{l,n}^*$  is an irreducible projective variety of dimension  $l(n-l)$  with homogeneous coordinate ring isomorphic to the subalgebra

$$A^* := \mathbb{C}[t_I^* \mid I \in P_l(n)],$$

where

$$(1.3.10) \quad t_I^* := \bar{\xi}_{\{n-l+1, \dots, n\}}^I, \quad (I \in P_l(n))$$

are the *dual Plücker coordinates*.

The map  $*$  :  $\xi_j^i \mapsto \bar{\xi}_j^i$ ,  $*$  :  $\det^{-1} \mapsto \det$  extends uniquely to a anti-linear involution  $*$  of  $\mathbb{C}[G]$ , which maps  $A$  bijectively onto  $A^*$ . In fact,  $*$  maps the Plücker coordinate  $t_I$  to the dual Plücker coordinate  $t_I^*$  for all  $I \in P_l(n)$ . Hence the algebraic structure of  $A^*$  is determined by the algebraic structure of  $A$ .

A so-called real form  $U$  of  $G$  is associated with the involution  $*$  on  $\mathbb{C}[G]$  by

$$U := \{g \in G \mid \overline{f(g)} = f^*(g) \quad \forall f \in \mathbb{C}[G]\}.$$

Here  $U$  is the group consisting of the  $n$  by  $n$  unitary matrices.



Observe that the isotropy subgroup  $G_x^* := \{g \in G \mid g.[x^*] = [x^*]\}$  of  $[x^*]$  is given by

$$G_x^* := \{g = (g_{ij})_{i,j} \in G \mid g_{ij} = 0 \text{ for } n-l+1 \leq i \leq n, 1 \leq j \leq n-l\},$$

which is a parabolic subgroup of  $G$  since it contains the subgroup of upper triangular matrices. Observe furthermore that the Levi subgroup  $L$  is equal to the intersection  $G_x \cap G_x^*$ .

Now we can relate the homogeneous coordinate rings  $A$  and  $A^*$  with the subalgebra  $\mathbb{C}[G/L]$  (1.3.9) as follows. Consider the diagonal action of  $G$  on  $\Lambda^l(V) \otimes \Lambda^l(V^*)$ , which is defined by  $g.(v \otimes w) := g.v \otimes g.w$  for  $g \in G$ ,  $v \in \Lambda^l(V)$  and  $w \in \Lambda^l(V^*)$ . The diagonal action descends to a well-defined action on  $\mathbb{P}(\Lambda^l(V) \otimes \Lambda^l(V^*))$ , cf. (1.3.4). Let  $B$  be the homogeneous coordinate ring of the closure of the orbit  $G.[y]$ , where  $y$  is given by

$$(1.3.11) \quad y := v_{\{n-l+1, \dots, n\}} \otimes v_{\{n-l+1, \dots, n\}}^* \in \Lambda^l(V) \otimes \Lambda^l(V^*).$$

The algebra  $B$  can be identified with the subalgebra of  $\mathbb{C}[G]$  generated by the elements  $t_I t_J^*$  ( $I, J \in P_l(n)$ ) via the dual  $\psi^*$  of the morphism  $\psi : g \mapsto g.y$ . On the other hand,  $L$  stabilizes  $y$ , hence the image of  $\psi^*$  is contained in  $\mathbb{C}[G/L]$ . It can be shown that  $\mathbb{C}[G/L]$  is in fact generated by the elements  $t_I t_J^*$  ( $I, J \in P_l(n)$ ), i.e. any right  $L$ -invariant polynomial function on  $G$  is a polynomial in the elements  $t_I t_J^*$  ( $I, J \in P_l(n)$ ). This result can be informally restated as follows: any right  $L$ -invariant polynomial function on  $G$  is a sum of products of a holomorphic polynomial and an anti-holomorphic polynomial in the Plücker coordinates  $t_I$  ( $I \in P_l(n)$ ). From this viewpoint we have thus obtained a factorization of the algebra  $\mathbb{C}[G/L]$  in terms of algebras of holomorphic and anti-holomorphic polynomials.

## 1.4. Overview of the remaining chapters

**1.4.1. Multivariable orthogonal polynomials.** The study of orthogonal polynomials related to multivariable beta type integrals started in the 1970's with the work of James and Constantine [45], Vretare [131], [132] and Koornwinder and Sprinkhuizen-Kuyper [61], [62] [69], [113]. In the late 1980's Heckman and Opdam [36], [37], [38] associated to each irreducible root system certain multivariable analogues of the Jacobi polynomials. Many important properties of the polynomials were derived, such as orthogonality relations, quadratic norm evaluations and the existence of a "large enough" system of differential equations for which the Jacobi polynomials are joint eigenfunctions; the so-called hypergeometric differential equations.

Macdonald [82] introduced  $q$ -deformations of the Heckman-Opdam polynomials and proved orthogonality relations with respect to multivariable continuous  $q$ -beta type integrals. Cherednik's affine Hecke-algebraic approach [12], [13], [14], [15] has led to a good understanding of the basic properties of the Macdonald polynomials.

The Macdonald polynomials associated to the non-reduced root system BC of rank 1 form a two parameter subfamily of the four parameter family of Askey-Wilson polynomials. The Askey-Wilson polynomials are the orthogonal polynomials with respect to the

continuous  $q$ -beta integral (1.2.5). Koornwinder [65] extended the definition of the  $BC$  type Macdonald polynomials to a five parameter family of orthogonal polynomials, which for rank 1 reduce to the full four parameter family of Askey-Wilson polynomials. Similar results as mentioned above for the Heckman-Opdam polynomials and the Macdonald polynomials have been proved for the Koornwinder polynomials by work of Koornwinder [65], van Diejen [17], [18], Noumi [91], Macdonald [86] and Sahi [106].

The Askey-Wilson polynomials are on top of the so-called Askey tableau. The Askey tableau is a hierarchy of families of basic hypergeometric orthogonal polynomials which are joint eigenfunctions of a second order  $q$ -difference operator. Certain families can be obtained from other families by limit transitions or specializations of the parameters, which induces the hierarchy structure between the families in the Askey tableau. The example treated in Section 1.2 is directly related to the hierarchy structure between the Askey-Wilson polynomials and the orthogonal polynomials associated to the discrete  $q$ -beta integral (1.2.3), which are called the little  $q$ -Jacobi polynomials. More explicitly, in Section 1.2 it is mentioned that the discrete  $q$ -beta integral (1.2.3) is a limit case of the Askey-Wilson integral (1.2.5). The corresponding limit on the level of orthogonal polynomials gives the hierarchy structure between Askey-Wilson polynomials and the little  $q$ -Jacobi polynomials.

In Chapter 2 and Chapter 3 multivariable analogues of three families of orthogonal polynomials in the Askey tableau are introduced, namely the  $q$ -Racah polynomials and the big and little  $q$ -Jacobi polynomials. It is also proved that limit transitions between these three families and the Koornwinder polynomials exist, which can be seen as a multivariable generalization of the hierarchy structure in the Askey tableau. Furthermore, full orthogonality of the polynomials is established with respect to multivariable  $q$ -beta type integrals. Their quadratic norms are computed and it is shown that for each family there is a second order  $q$ -difference operator which is diagonalized by the orthogonal polynomials. The multivariable big and little  $q$ -Jacobi polynomials satisfy orthogonality relations with respect to multivariable discrete  $q$ -beta integrals which have been introduced in 1980 by Askey [5], and have been studied intensively thereafter in several papers, e.g. [5], [33], [54], [4], [29], [127].

An important tool for obtaining the above mentioned results on the multivariable Askey tableau is the development of a residue calculus for Gustafson's multidimensional analogue of the continuous  $q$ -beta integral (1.2.5), which is presented in Chapter 2. Discrete multivariable  $q$ -analogues of the beta integral can now be obtained from Gustafson's integral by suitable limit transitions. This was shown for a special one variable example in Section 1.2. Several properties of the limit cases of the Koornwinder polynomials are proved by taking the limits in the corresponding results for the Koornwinder polynomials.

The results of Chapter 2 and Chapter 3 have appeared in several papers. The multivariable big and little  $q$ -Jacobi polynomials were introduced and studied in [116]. Several limit transitions between multivariable orthogonal polynomials were studied in an algebraic manner, cf. Section 3.6, in the joint paper [121] with Koornwinder. The multivariable  $q$ -Racah polynomials were introduced and studied in the joint paper [20] with van

Diejen. In the papers [117], [118] the Koornwinder polynomials for special parameter values are studied with respect to partially discrete orthogonality measures. The general residue calculus for Koornwinder polynomials has appeared in [119]. In Chapter 2 and Chapter 3 we have mainly followed the approach of [119]. Finally, the theory in Chapter 2 and Chapter 3 in the one variable case has appeared in the joint paper [122] with Koornwinder.

**1.4.2. Multivariable orthogonal polynomials and quantum Grassmannians.** In the mid 1980's Drinfeld [27] and Jimbo [47] quantized the universal enveloping algebra  $U(\mathfrak{g})$  of a simple Lie algebra  $\mathfrak{g}$ . They obtained the “standard” quantized universal enveloping algebra  $U_h(\mathfrak{g})$ , which is a Hopf-algebra quantization of  $U(\mathfrak{g})$  endowed with the co-Poisson structure induced by the standard solution  $r$  of the modified classical Yang-Baxter equation. The defining relations of  $U_h(\mathfrak{g})$  are given by the so-called quantized Serre relations. In particular, the definition of  $U_h(\mathfrak{g})$  depends on a particular choice of a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  and of a choice of simple roots  $\Delta = \{\alpha_1, \dots, \alpha_r\}$  for the root system associated with  $(\mathfrak{g}, \mathfrak{h})$ .

The Hopf-algebra dual  $\mathbb{C}_h[G]$  of  $U_h(\mathfrak{g})$  is a quantization of the function algebra of regular functions on the connected, simply connected, simple Lie group  $G$  with Lie algebra  $\mathfrak{g}$ . The associated Poisson-Lie group structure on  $G$  in the semi-classical limit is given by the so-called Sklyanin-bracket on  $G$  associated with  $r$ . This bracket is also known as the Bruhat-Poisson bracket on  $G$ .

In Chapter 4,5 and 6 of the Tract an important object of study is a rational form of  $\mathbb{C}_h[G]$ , which is denoted here by  $\mathbb{C}_q[G]$ . Here  $q$  is assumed to be specialized to a value in the open interval  $(0, 1)$ . For several simple Lie groups  $G$ , there is an explicit realization of  $\mathbb{C}_q[G]$  in terms of generators and relations. For the purpose of this section, we give here the construction of  $\mathbb{C}_q[G]$  for the reductive Lie group  $G = GL(n, \mathbb{C})$  in terms of generators and relations. The quantized function algebra  $\mathbb{C}_q[SL(n, \mathbb{C})]$  is then an Hopf subgroup of  $\mathbb{C}_q[GL(n, \mathbb{C})]$ , which can formally be obtained from  $\mathbb{C}_q[GL(n, \mathbb{C})]$  by setting the quantum determinant equal to 1. The algebra  $\mathbb{C}_q[GL(n, \mathbb{C})]$  is generated by the elements  $t_{ij}$  ( $1 \leq i, j \leq n$ ) and  $\det_q^{-1}$ , subject to the relations

$$\begin{aligned} t_{kit}t_{kj} &= qt_{kj}t_{ki}, & t_{ik}t_{jk} &= qt_{jk}t_{ik} \quad (i < j), \\ t_{ilt}t_{kj} &= t_{kj}t_{il}, & t_{ij}t_{kl} - t_{kl}t_{ij} &= (q - q^{-1})t_{il}t_{kj} \quad (i < k, j < l), \end{aligned}$$

and  $\det_q^{-1}$  is defined as the inverse of the quantum determinant

$$\det_q := \sum_{\sigma \in \mathfrak{S}_n} (-q)^{l(\sigma)} t_{\sigma(1)1} t_{\sigma(2)2} \cdots t_{\sigma(n)n}.$$

Observe the striking similarity with the coordinate ring  $\mathbb{C}[G]$  of  $G = GL(n, \mathbb{C})$  as considered in Section 1.3. In particular, it is immediate from the above description of  $\mathbb{C}_q[G]$  that the coordinate ring  $\mathbb{C}[G]$  is formally reobtained by taking the limit  $q \uparrow 1$ .

The function algebra  $\mathbb{C}_q[G/L]$  of the regularly embedded quantum Grassmannian is defined as the “standard” quantization of the algebra of functions  $\mathbb{C}[G/L]$ , where  $G =$

$GL(n, \mathbb{C})$ ,  $L = GL(n-l, \mathbb{C}) \times GL(l, \mathbb{C})$  and  $\mathbb{C}[G/L]$  is the subalgebra of right  $L$ -invariant functions in  $\mathbb{C}[G]$ , cf. (1.3.9). The quantized function algebra  $\mathbb{C}_q[G/L]$  can be alternatively described as the subalgebra of right  $\mathbb{C}_q[L]$ -invariant elements in  $\mathbb{C}_q[G]$ , where  $\mathbb{C}_q[L]$  is the obvious quantum analogue of the subgroup  $L = GL(n-l, \mathbb{C}) \times GL(l, \mathbb{C})$ , cf. (1.3.8). This quantization of the Grassmann manifold is called regularly embedded because the quantum subgroup  $\mathbb{C}_q[L]$  contains the quantized diagonal matrices. This property turns out to have vast implications for the associated harmonic analysis. For instance, it implies that the associated zonal spherical functions satisfy orthogonality relations with respect to completely discrete orthogonality measures.

In Chapter 4 it is shown that the zonal spherical functions on the regularly embedded quantum Grassmannian can in fact be identified with multivariable big and little  $q$ -Jacobi polynomials. The strategy for obtaining this result is as follows. Noumi, Dijhuizen and Sugitani [92] introduced a one parameter family of quantum complex Grassmannians and they identified the associated zonal spherical functions with a subfamily of the Koornwinder polynomials. The regularly embedded quantum Grassmannian can be formally reobtained from the one parameter family of quantum Grassmannians by sending the parameter to infinity. In Chapter 4 it is shown that such limits on the level of quantum Grassmannians correspond on the level of zonal spherical functions with the limits from Koornwinder polynomials to multivariable big and little  $q$ -Jacobi polynomials as studied in Chapter 2. The results of Chapter 1 and Chapter 2 therefore play an essential role in the study of the limit transitions on the one parameter family of quantum Grassmannians.

The rank 1 case of these results was obtained by Koornwinder [64], [66] for 2-spheres and by Dijkhuizen and Noumi [24]. Chapter 4 is based on the joint paper [25] with Dijkhuizen. The results in Section 4.5 and Section 4.6 have appeared before in the paper [92] of Noumi, Dijkhuizen and Sugitani without proofs.

**1.4.3. Quantum Plücker coordinates and their generalizations.** The concept of Plücker coordinates on Grassmannians can be generalized to arbitrary flag manifolds. The general definition is based on the characterization of the Plücker coordinates as matrix coefficients of a finite dimensional (irreducible) representation, cf. (1.3.2), (1.3.6).

Flag manifolds are defined as the homogeneous spaces of the form  $G/P$ , where  $G$  is a connected, simply connected, simple Lie group and  $P$  is a parabolic subgroup of  $G$ . In this section the parabolic subgroup  $P$  is assumed to be standard with respect to the fixed Cartan subalgebra  $\mathfrak{h}$  and with respect to the fixed simple roots  $\Delta$ . The standard parabolic subgroups are naturally parametrized by subsets of  $\Delta$ .

On the other hand, irreducible finite dimensional representations of  $G$  are parametrized by an integral cone  $\bigoplus_{\alpha \in \Delta} \mathbb{Z}_+ \varpi_\alpha$ , the so-called integral cone of dominant integral weights. The  $\varpi_\alpha$  ( $\alpha \in \Delta$ ) are called the fundamental dominant weights and form a type of dual basis with respect to the simple roots  $\Delta$ . For a parabolic subgroup  $P$  of  $G$  corresponding to a particular subset  $I \subset \Delta$ , the (generalized) Plücker coordinates can be defined as certain matrix coefficients of the finite dimensional irreducible representations of  $G$  corresponding to the fundamental weights  $\varpi_\alpha$  ( $\alpha \in I$ ). Dual (generalized) Plücker coordinates are then defined as certain matrix coefficients of the corresponding

dual representations. Quantum analogues of the generalized Plücker coordinates and of the generalized dual Plücker coordinates can be defined in a similar manner by using the finite dimensional corepresentation theory of  $\mathbb{C}_q[G]$ .

If  $L$  is the Levi component of the parabolic subgroup  $P$ , then the “standard” quantized function algebra  $\mathbb{C}_q[G/L]$  can be defined as the subalgebra of right  $\mathbb{C}_q[L]$ -invariant elements of the quantized function algebra  $\mathbb{C}_q[G]$ , where  $\mathbb{C}_q[L]$  is the quantum subgroup associated with the Levi component  $L$ . Then, products of (generalized) quantum Plücker coordinates and dual (generalized) quantum Plücker coordinates lie in the quantized function algebra  $\mathbb{C}_q[G/L]$ . From an informal point of view, this means that  $\mathbb{C}_q[G/L]$  contains the quantized algebra of zero-weighted complex-valued polynomial functions on  $G/L$ .

One of the questions which is addressed in Chapter 5 and Chapter 6 is the following: Is  $\mathbb{C}_q[G/L]$  generated as algebra by the products of the quantum Plücker coordinates and the dual quantum Plücker coordinates? If the answer is affirmative, then the quantized function algebra  $\mathbb{C}_q[G/L]$  is called factorizable. Affirmative answers to this factorization problem is given for an interesting class of flag manifolds. In fact it turns out that the answer is affirmative for *all* flag manifolds (unpublished result of the author).

In Chapter 5 the factorization problem is proved for the regularly embedded quantum Grassmannian. Furthermore, the defining relations between the quantum Plücker coordinates and the dual quantum Plücker coordinates are derived in case of the complex Grassmannian. To arrive at this result, it is important to understand the algebraic structure of the algebra  $A_q$ , which is by definition the algebra generated by the quantum Plücker coordinates. The algebra  $A_q$  is the quantized homogeneous coordinate ring of the Grassmann manifold  $Y_{l,n}$ , cf. Section 1.3. The algebraic structure of  $A_q$  is well understood due to the work of Taft and Towber [126] and of Noumi, Yamada and Mimachi [96]. In [126] it was shown that the defining relations of the quantum Plücker coordinates are given by Young symmetry relations, or equivalently by  $q$ -Garnir relations. Both of these relations contain the quantum analogues of the Plücker relations (1.3.7) as special cases. The results of Chapter 5 are based on a handwritten manuscript of the author.

In Chapter 6 the factorization problem is considered for arbitrary flag manifolds. The factorization is proved for an interesting class of flag manifolds, which contains in particular the irreducible compact Hermitean symmetric spaces. This part is based on the joint paper [123] with Dijkhuizen.

**1.4.4. Quantum orbit method for flag manifolds.** In Section 1.3 of this chapter an involution on the coordinate ring  $\mathbb{C}[G]$  of  $G = GL(n, \mathbb{C})$  was introduced. This involution corresponds on the level of groups with choosing the  $n$  by  $n$  unitary matrices as real form of  $G$ . The subalgebra  $\mathbb{C}[G/L]$  (1.3.9) of right  $L = GL(n-l, \mathbb{C}) \times GL(l, \mathbb{C})$  invariant polynomial functions on  $G = GL(n, \mathbb{C})$  is stable under this involution. In fact, the involution maps the Plücker coordinates to the dual Plücker coordinates. There is a natural anti-linear anti-involution  $*$  of the standard quantized function algebra  $\mathbb{C}_q[G]$  which, in the semi-classical limit, reduces to choosing the  $n$  by  $n$  unitary matrices as real form of  $G = GL(n, \mathbb{C})$ . Again,  $\mathbb{C}_q[G/L]$  is  $*$ -stable and the anti-involution maps the quantum Plücker coordinates to the dual quantum Plücker coordinates.

For a given connected, simply connected, simple Lie group  $G$  with Lie algebra  $\mathfrak{g}$  and Cartan subalgebra  $\mathfrak{h}$  there is a standard way to construct a compact real form  $U$  of  $G$  using a special root basis of  $\mathfrak{g}$ , cf. [40, Theorem 6.3]. The compact real form  $U$  has the property that for any standard parabolic subgroup  $P$ ,  $K = P \cap U$  is a compact real form of the Levi component  $L$  of  $P$ . Furthermore,  $G/P$  is isomorphic to  $U/K$  as a real manifold.

There exists an anti-linear anti-involution  $*$  on the quantized function algebra  $\mathbb{C}_q[G]$  which corresponds in the semi-classical limit to the above mentioned choice of compact real form  $U$ . To emphasize that  $\mathbb{C}_q[G]$  is considered with this  $*$ -structure, it is customary to write  $\mathbb{C}_q[U]$  instead of  $\mathbb{C}_q[G]$ . The subalgebra  $\mathbb{C}_q[G/L]$  with  $L$  the Levi component of a standard parabolic subgroup  $P$  is stable under the  $*$ -involution. Furthermore, the  $*$ -involution maps quantum Plücker coordinates to dual quantum Plücker coordinates. In Chapter 6, the algebra  $\mathbb{C}_q[G/L]$  is considered with the above mentioned  $*$ -structure. To emphasize the choice of  $*$ -structure, we write  $\mathbb{C}_q[U/K]$  ( $K := P \cap U$ ) for the algebra  $\mathbb{C}_q[G/L]$  with this particular choice of  $*$ -structure.

In Chapter 6 the quantum orbit method is developed for flag manifolds  $U/K$  by relating the irreducible  $*$ -representations of the  $*$ -algebra  $\mathbb{C}_q[U/K]$  to the geometry of the underlying Poisson structure on  $U/K$ . Since  $\mathbb{C}_q[U/K]$  is a  $*$ -subalgebra of  $\mathbb{C}_q[U]$ , the Poisson structure of  $U/K$  in the semi-classical limit is the induced Bruhat-Poisson structure of  $U$ , i.e. it is the unique Poisson structure on  $U/K$  such that the natural projection  $U \rightarrow U/K$  preserves the Poisson structures.

The Poisson geometry of  $U$  and  $U/K$  with respect to the Bruhat-Poisson bracket was studied by Soibelman [110], respectively Lu and Weinstein [80]. It was shown that the symplectic foliation of  $U$  is a refinement of the Bruhat decomposition of  $U$  and that the symplectic foliation of  $U/K$  coincides with the Schubert cell decomposition of  $U/K$ . Every symplectic leaf of  $U$  is mapped surjectively onto a Schubert cell of  $U/K$  under the natural projection  $U \rightarrow U/K$ .

The irreducible  $*$ -representations of the standard quantized function algebra of the group of  $n$  by  $n$  unitary matrices were classified independently by Koelink [60] and Soibelman [109]. In [110], Soibelman classified the irreducible  $*$ -representations of  $\mathbb{C}_q[U]$  for an arbitrary compact simple Lie group  $U$ . Soibelman showed that the irreducible  $*$ -representations of  $\mathbb{C}_q[U]$  are naturally parametrized by the symplectic leaves of the Bruhat-Poisson structure on  $U$ .

The natural embedding  $\mathbb{C}_q[U/K] \hookrightarrow \mathbb{C}_q[U]$  can be interpreted as the quantized dual of the natural projection  $U \rightarrow U/K$ . In Chapter 6 it is shown there is a close connection between the properties of the natural projection  $U \rightarrow U/K$  and the properties of its quantized dual. For instance, it is shown that an irreducible  $*$ -representation  $\pi$  of  $\mathbb{C}_q[U]$  remains irreducible as  $*$ -representation of  $\mathbb{C}_q[U/K]$  if and only if the symplectic leaf corresponding to  $\pi$  is mapped isomorphically onto its image under the natural projection  $U \rightarrow U/K$ . Since for each Schubert cell  $Y$  of  $U/K$  there exist symplectic leaves of  $U$  which are mapped isomorphically onto  $Y$ , we thus obtain in a natural way an irreducible  $*$ -representation of  $\mathbb{C}_q[U/K]$  for every symplectic leaf  $Y$  of  $U/K$ . Irreducible  $*$ -representations corresponding to different Schubert cells turn out to be inequivalent.

A  $*$ -representation of a factorizable quantized function algebras  $\mathbb{C}_q[U/K]$  is completely determined by its action on the products of the quantum Plücker coordinates and the dual quantum Plücker coordinates. This fact is used to give a complete classification of the irreducible  $*$ -representations of a factorizable quantized function algebra  $\mathbb{C}_q[U/K]$ . For factorizable quantized function algebras  $\mathbb{C}_q[U/K]$  it is shown that the irreducible  $*$ -representations of  $\mathbb{C}_q[U/K]$  obtained by restriction of irreducible  $*$ -representations of  $\mathbb{C}_q[U]$  exhaust the equivalence classes of irreducible  $*$ -representations, i.e. the equivalence classes of irreducible  $*$ -representations are parametrized by the Schubert cells of  $U/K$ . In particular, the complete classification of the irreducible  $*$ -representations is obtained for the quantized function algebras of the irreducible compact Hermitean symmetric spaces.

These and other connections between the Poisson geometry of flag manifolds and the irreducible  $*$ -representations of the quantized function algebras of flag manifolds are explored in full detail in Chapter 6.

Chapter 6 is based on the joint paper [123] with Dijkhuizen.

### 1.5. Notations and conventions

The following notations and conventions are used throughout the Tract.

- $\mathbb{N} := \{1, 2, 3, \dots\}$  and  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ .
- Sums over empty index sets are equal to 0, products over empty index sets are equal to 1.
- $q$  is a fixed real number in the open interval  $(0, 1)$ , unless explicitly stated otherwise.
- $[k, l] := \{k, k+1, \dots, l-1, l\}$  for integers  $k, l$  with  $k < l$ .
- An associative algebra is by definition an associative algebra with unit.
- The  $q$ -shifted factorial is defined by

$$(1.5.1) \quad (a; q)_b := \frac{(a; q)_\infty}{(aq^b; q)_\infty}, \quad (a; q)_\infty := \prod_{j=0}^{\infty} (1 - q^j a),$$

provided that  $aq^b \notin \{q^{-k}\}_{k \in \mathbb{Z}_+}$ . For  $b = k \in \mathbb{Z}_+$ , this reduces to  $(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i)$ , which is well defined for all  $a \in \mathbb{C}$ . The notation

$$(a_1, \dots, a_m; q)_b := \prod_{j=1}^m (a_j; q)_b$$

is used for products of  $q$ -shifted factorials.





## A residue calculus for Koornwinder polynomials

### 2.1. Introduction

Koornwinder [65] introduced a five parameter multivariable generalization of the four parameter family of Askey-Wilson polynomials [7] and showed that the polynomials satisfy orthogonality relations with respect to the absolutely continuous measure of Gustafson's multidimensional Askey-Wilson integral [31]

$$(2.1.1) \quad \frac{1}{(2\pi i)^n} \iint_{z \in T^n} \prod_{\substack{1 \leq k < l \leq n \\ \varepsilon_k, \varepsilon_l \in \{\pm 1\}}} \frac{(z_k^{\varepsilon_k} z_l^{\varepsilon_l}; q)_\infty}{(t z_k^{\varepsilon_k} z_l^{\varepsilon_l}; q)_\infty} \prod_{i=1}^n \frac{(z_i^2, z_i^{-2}; q)_\infty}{\prod_{j=0}^3 (t_j z_i, t_j z_i^{-1}; q)_\infty} \frac{dz_i}{z_i} \\ = 2^n n! \prod_{i=1}^n \frac{(t, t^{2n-i-1} t_0 t_1 t_2 t_3; q)_\infty}{(q, t^{n-i+1}; q)_\infty \prod_{0 \leq k < l \leq 3} (t^{n-i} t_k t_l; q)_\infty},$$

where  $T^n$  is the  $n$ -torus in  $C^n$  and  $|t_i| < 1$ ,  $t \in (0, 1)$ . This five parameter family of multivariable orthogonal polynomials is nowadays known as the family of Koornwinder polynomials. Certain three parameter subfamilies of the Koornwinder polynomials coincide with the families of BC type Macdonald polynomials, which were previously introduced and studied by Macdonald [86]. Van Diejen [16] has shown that the Koornwinder polynomials contain all families of Macdonald polynomials associated with the classical root systems as special cases or, in the case of root system  $A$ , as the highest homogeneous part.

Several important properties of the Koornwinder polynomials were derived in the past years. We mention here two properties which will play an important role in this chapter. Koornwinder [65] introduced an explicit second order  $q$ -difference operator which simultaneously diagonalizes the Koornwinder polynomials. Van Diejen [17] evaluated the quadratic norms for a four parameter family of the Koornwinder polynomials using so-called Pieri formulas. By recent results of Sahi [106], van Diejen's quadratic norm evaluations are valid for the full five parameter family of Koornwinder polynomials, see also [120].

In [65], [17] the orthogonality relations and norm evaluations with respect to the measure associated with (2.1.1) were derived for parameters  $(\underline{t}, t)$  with  $|t_i| \leq 1$ . In Section 2.3 we extend these results to the case that the four parameters  $\underline{t}$  are generically complex. Intuitively, the results for  $\underline{t}$  generically complex follow from the results for  $|t_i| \leq 1$  by analytic continuation, after replacing the torus  $T^n$  in the orthogonality measure by a suitable Cartesian product  $C^n$  of a Jordan curve  $C$ . Informally rephrased, the results in Section

2.3 follow from a multidimensional analytic version of Cauchy's Theorem for the measure associated with (2.1.1), analytic in the sense that no poles of the measure are picked up. The technicalities are a little bit more involved and will be discussed in Section 2.3.

In Section 2.4 a multidimensional meromorphic version of Cauchy's Theorem is developed for the measure associated with (2.1.1), so that certain poles of the measure can be picked up by shifting the multidimensional contour. The completely discrete weights which arise from this residue calculus contain a common factor which has a pole when the parameters satisfy a type of truncation condition. By dividing out this common factor, the partly continuous weights are zero for parameters satisfying the truncation condition, and all that remains is a discrete, finite orthogonality measure. The Koornwinder polynomials with parameters satisfying this truncation condition are multivariable analogues of the well-known family of one variable  $q$ -Racah polynomials. These multivariable analogues of the  $q$ -Racah polynomials are studied in Section 2.5.

In Section 2.6 orthogonality relations for the Koornwinder polynomials are derived with respect to positive, partly discrete orthogonality measures. The results in Section 2.6 are important for the understanding of limit transitions from Koornwinder polynomials to multivariable analogues of big and little  $q$ -Jacobi polynomials, which will be the main subject of the next chapter.

This chapter is started with a short review on the theory of one variable Askey-Wilson polynomials and one variable  $q$ -Racah polynomials.

## 2.2. One variable Askey-Wilson and $q$ -Racah polynomials

Following the notations in [30] we define the basic hypergeometric series  ${}_r\phi_s$  by

$$(2.2.1) \quad {}_r\phi_s \left( \begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) = \sum_{m=0}^{\infty} \frac{(a_1, \dots, a_r; q)_m}{(b_1, \dots, b_s, q; q)_m} \left( (-1)^m q^{\binom{m}{2}} \right)^{1+s-r} z^m.$$

A set of orthogonal polynomials is called of basic hypergeometric type if the orthogonal polynomials can be expressed in terms of the basic hypergeometric series. Askey and Wilson [7] introduced a very general family of basic hypergeometric orthogonal polynomials depending on four parameters  $\underline{t} = (t_0, t_1, t_2, t_3)$  which is nowadays known as the family of Askey-Wilson polynomials. In terms of the basic hypergeometric series (2.2.1) they are given by

(2.2.2)

$$P_n(z; \underline{t}) := \frac{(t_0 t_1, t_0 t_2, t_0 t_3; q)_n}{t_0^n (t_0 t_1 t_2 t_3 q^{n-1}; q)_n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, q^{n-1} t_0 t_1 t_2 t_3, t_0 z, t_0 z^{-1} \\ t_0 t_1, t_0 t_2, t_0 t_3 \end{matrix}; q, q \right)$$

for  $n \in \mathbb{Z}_+$ . Observe that  $P_n(z)$  is a monic polynomial in  $z + z^{-1}$  of degree  $n$ .

The orthogonality relations and norm evaluations for the monic Askey-Wilson polynomials can be stated as follows.

**THEOREM 2.2.1.** ([7, Theorem 2.3]) *Assume that pairwise products of  $t_0, t_1, t_2, t_3$  as a multiset (so both  $t_0^2$  and  $t_0 t_1$  are considered among the products) do not belong to*

the set  $\{q^{-j}\}_{j \in \mathbb{Z}_+}$ . Then the monic Askey-Wilson polynomials satisfy the orthogonality relations

$$\frac{1}{2\pi i} \int_{z \in C} (P_m P_n)(z; \underline{t}) w_c(z; \underline{t}) \frac{dz}{z} = \delta_{m,n} \mathcal{N}(n; \underline{t})$$

with weight function

$$(2.2.3) \quad w_c(x; \underline{t}) := \frac{(x^2, x^{-2}; q)_\infty}{(t_0 x, t_0 x^{-1}, t_1 x, t_1 x^{-1}, t_2 x, t_2 x^{-1}, t_3 x, t_3 x^{-1}; q)_\infty}.$$

Here  $C$  is a positively oriented, continuous differentiable Jordan curve containing 0 and the four sequences  $\{t_i q^j\}_{j \in \mathbb{Z}_+}$  ( $i \in [0, 3]$ ) and separating them from  $\{t_i^{-1} q^{-j}\}_{j \in \mathbb{Z}_+}$  ( $i \in [0, 3]$ ). The quadratic norms  $\mathcal{N}(n)$  of the monic Askey-Wilson polynomials are explicitly given by

$$\mathcal{N}(n; \underline{t}) = \frac{2(q^{2n-1} t_0 t_1 t_2 t_3, q^{2n} t_0 t_1 t_2 t_3; q)_\infty}{(q^{n+1}, q^{n-1} t_0 t_1 t_2 t_3, q^n t_0 t_1, q^n t_0 t_2, q^n t_0 t_3, q^n t_1 t_2, q^n t_1 t_3, q^n t_2 t_3; q)_\infty}.$$

The theorem implies in particular that the monic Askey-Wilson polynomials  $P_n(z; \underline{t})$  are symmetric with respect to permutations of the four variables  $\underline{t}$  (which cannot be immediately read off from the explicit expressions (2.2.2)). For the proof of the orthogonality relations and norm evaluations, Askey and Wilson [7] used the  $q$ -Pfaff-Saalschütz sum [7, (1.29)], [30, (II.12), p. 237] and the explicit evaluation of the integral over the weight function,

$$(2.2.4) \quad \frac{1}{2\pi i} \int_{z \in C} w_c(z; \underline{t}) \frac{dz}{z} = \frac{2(t_0 t_1 t_2 t_3; q)_\infty}{(q, t_0 t_1, t_0 t_2, t_0 t_3, t_1 t_2, t_1 t_3, t_2 t_3; q)_\infty}$$

(cf. [7, Theorem 2.1]). The integral (2.2.4) is a  $q$ -analogue of the classical beta integral and its evaluation is proved in [7] by summing up four sequences of residues by a summation formula of a very-well poised  ${}_6\phi_5$  series [7, (2.2)], [30, (II.20), p.238] and subsequently summing the four remaining terms with the help of an elliptic function identity. More elementary proofs of (2.2.4) were obtained, for instance, in [103], [43] and [67].

A partly discrete orthogonality measure can be obtained by pulling  $C$  over some of the poles of  $w_c$ . The poles of  $w_c$  are simple for generic parameters  $\underline{t}$  and are given by the eight sequences  $\{t_i q^j\}_{j \in \mathbb{Z}_+}$ ,  $\{t_i^{-1} q^{-j}\}_{j \in \mathbb{Z}_+}$  ( $i \in [0, 3]$ ). We write

$$(2.2.5) \quad w_d(eq^i; e) = w_d(eq^i; e; f, g, h) := \operatorname{res}_{z=eq^i} \left( \frac{w_c(z; \underline{t})}{z} \right)$$

for the residues, where  $f, g, h$  are such that  $\{e, f, g, h\} = \{t_0, t_1, t_2, t_3\}$  (counted with multiplicity). When  $w_c$  has a simple pole in  $eq^i$ , then

$$(2.2.6) \quad \operatorname{res}_{z=e^{-1}q^{-i}} \left( \frac{w_c(z; \underline{t})}{z} \right) = -w_d(eq^i; e; f, g, h)$$

by the invariance of  $w_c(z)$  under the inversion  $z \mapsto z^{-1}$ , and we have the explicit formula

$$(2.2.7) \quad w_d(eq^i; e; f, g, h) := \frac{(e^{-2}; q)_\infty}{(q, ef, f/e, eg, g/e, eh, h/e; q)_\infty} \frac{(e^2, ef, eg, eh; q)_i}{(q, qe/f, qe/g, qe/h; q)_i} \frac{(1 - e^2 q^{2i})}{(1 - e^2)} \left( \frac{q}{efgh} \right)^i$$

(see [7, Theorem 2.4] or [30, (7.5.22)] to avoid a small misprint).

The monic  $q$ -Racah polynomials depend (apart from  $q$ ) on three continuous parameter and one discrete parameter  $N \in \mathbb{N}$ . They are the finite family

$$\{P_n(\cdot; \underline{t}_N; q) \mid n \in [0, N]\}$$

of Askey-Wilson polynomials for the special parameters  $\underline{t}_N := (t_0, t_1, t_2, t_0^{-1}q^{-N})$  where  $N \in \mathbb{N}$ . Since the parameters  $\underline{t}_N$  do no longer satisfy the assumptions of Theorem 2.2.1, it is convenient to think of the  $q$ -Racah polynomials as limit cases of the monic Askey-Wilson polynomials where the limit is given by sending  $t_3$  to  $t_0^{-1}q^{-N}$ .

The orthogonality relations and norm evaluations for the monic  $q$ -Racah polynomials can be stated as follows.

**THEOREM 2.2.2.** ([6, Section 2]) *Let  $N \in \mathbb{N}$ . For generic parameters  $t_0, t_1, t_2$  we have the orthogonality relations*

$$\sum_{i=0}^N (P_m P_n)(t_0 q^i; \underline{t}_N) \Delta^{qR}(t_0 q^i; \underline{t}_N) = \delta_{m,n} \mathcal{N}^{qR}(n; \underline{t}_N)$$

for  $m, n \in \{0, \dots, N\}$ , with

$$\Delta^{qR}(t_0 q^i; \underline{t}) := \frac{(1 - t_0^2 q^{2i})(t_0^2, t_0 t_1, t_0 t_2, t_0 t_3; q)_i}{(t_0 t_1 t_2 t_3 q^{-1})^i (1 - t_0^2)(q, qt_1^{-1} t_0, qt_2^{-1} t_0, qt_3^{-1} t_0; q)_i}$$

The quadratic norms of the monic  $q$ -Racah polynomials are explicitly given by

$$\mathcal{N}^{qR}(n; \underline{t}) := \frac{(q, t_0 t_1, t_0 t_2, t_0 t_3, t_1 t_2, t_1 t_3, t_2 t_3; q)_n (t_1/t_0, t_2/t_0, t_3/t_0, t_0 t_1 t_2 t_3; q)_\infty}{(q^{n-1} t_0 t_1 t_2 t_3; q)_n (t_0 t_1 t_2 t_3; q)_{2n} (t_1 t_2, t_1 t_3, t_2 t_3, t_0^{-2}; q)_\infty}$$

Askey and Wilson [6] obtained the orthogonality relations and norm evaluations for the  $q$ -Racah polynomials from a summation formula for very well poised terminating  ${}_6\phi_5$  series [6, (2.3)], [30, (II.21), p.238] and the  $q$ -Pfaff-Saalschütz sum [6, (2.5)], [30, (II.12), p.237]. In particular, they obtained the summation formula

$$(2.2.8) \quad \sum_{i=0}^N \Delta^{qR}(t_0 q^i; \underline{t}_N) = \frac{(qt_0^2, q/t_1 t_2; q)_N}{(qt_0/t_1, qt_0/t_2; q)_N}$$

using a summation formula for very well poised terminating  ${}_6\phi_5$  series [6, (2.3)], [30, (II.21), p.238].

It turn out that the orthogonality relations and norm evaluations for the  $q$ -Racah polynomials can be obtained by taking suitable limits in the orthogonality relations and norm

evaluations for the Askey-Wilson polynomials. The main trick here is to rescale a suitable partly discrete orthogonality measure for the Askey-Wilson polynomials such that certain common poles of the discrete weights become zeros for the continuous part of the orthogonality measure. These zeros cause the vanishing of the continuous part of the orthogonality measure in the limit  $\underline{t} \rightarrow \underline{t}_N$  (see [122] for the details). In Section 2.5 this approach is used to generalize Theorem 2.2.2 to the multivariable setting.

### 2.3. Koornwinder polynomials for generic parameters values

In this section the multivariable analogue of Theorem 2.2.1 is given. We first need to introduce some notations and conventions which will be used throughout this chapter. The number  $n \in \mathbb{N}$  denotes the number of independent variables of the multivariable polynomials under consideration. The  $n$  variables are denoted by  $z = (z_1, \dots, z_n)$ . A function  $h(z)$  for which a definition is given with respect to the  $n$  variables  $z = (z_1, \dots, z_n)$ , should be read with  $n = m$  if it follows from the context that  $z \in \mathbb{C}^m$ . If  $h(z)$  appears in formulas with  $z \in \mathbb{C}^m$  and  $m = 0$ , then  $h(z)$  should be read as 1.

Let  $\mathfrak{S} = \mathfrak{S}_n$  be the group of permutations of the set  $[1, n]$  and  $\mathcal{W} = \mathfrak{S} \ltimes \{\pm 1\}^n$  the Weyl group of type  $BC_n$ .  $\mathcal{W}$  acts by permutations and inversions on the algebra  $A := \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ . Denote  $A^{\mathcal{W}}$  for the subalgebra of  $\mathcal{W}$ -invariant functions in  $A$ . A basis for  $A^{\mathcal{W}}$  is given by the monomials  $\{m_\lambda \mid \lambda \in \Lambda\}$ , where  $\Lambda := \{\lambda \in \mathbb{Z}_+^n \mid \lambda_1 \geq \dots \geq \lambda_n\}$ , and

$$m_\lambda(z) := \sum_{\mu \in \mathcal{W}\lambda} z^\mu$$

with  $z^\mu = z_1^{\mu_1} \dots z_n^{\mu_n}$ . The  $\mathcal{W}$ -orbit of  $\lambda \in \Lambda \subset \mathbb{Z}^n$  is with respect to the natural action of  $\mathcal{W}$  on  $\mathbb{Z}^n$ .

Set  $\underline{t} = (t_0, t_1, t_2, t_3)$  for the four tuple of parameters  $t_0, t_1, t_2, t_3$ . We assume in this section that  $\underline{t} \in V$ , where  $V \subset \mathbb{C}^4$  is the following parameter domain.

DEFINITION 2.3.1. *Let  $V$  be the set of parameters  $\underline{t} \in (\mathbb{C}^*)^4$  for which*

$$\#\{\arg(t_i), \arg(t_i^{-1}) \mid i \in [0, 3]\} = 8$$

*and for which  $t_0 t_1 t_2 t_3 \notin \mathbb{R}_{\geq 1}$ . Here  $\arg(u) \in [0, 2\pi)$  is the argument of  $u \in \mathbb{C}^*$  and  $\mathbb{R}_{\geq 1} := \{r \in \mathbb{R} \mid r \geq 1\}$ .*

The measures which will be considered in this chapter will have their support on certain deformed  $n$ -tori  $C^n \subset \mathbb{C}^n$ , where  $C \subset \mathbb{C}$  are the following type of deformations of the unit circle  $T$ .

DEFINITION 2.3.2. *A continuous rectifiable Jordan curve  $C = \phi_C([0, 1]) \subset \mathbb{C}$  is called a deformed circle if  $C$  has a parametrization  $\phi_C$  of the form*

$$(2.3.1) \quad \phi_C(x) = r_C(x)e^{2\pi i x} \quad (x \in [0, 1]), \quad r_C : [0, 1] \rightarrow (0, \infty)$$

*and if  $C$  is invariant under inversion, i.e.  $C^{-1} := \{z^{-1} \mid z \in C\} = C$ . For  $\underline{t} \in V$ , a deformed circle  $C$  is called a  $\underline{t}$ -contour if the four parameters  $t_0, t_1, t_2, t_3$  are in the interior of  $C$ .*

For a deformed circle  $C = \phi_C([0, 1])$  the radial function  $r_C$  satisfies  $r_C(1-x) = (r_C(x))^{-1}$  since  $C = C^{-1}$ . Since a deformed circle  $C$  is by definition a closed contour, we furthermore have that  $r_C(0) = r_C(1/2) = r_C(1) = 1$ . Observe that the unit circle  $T$  is a deformed circle with  $r_T \equiv 1$ . If  $\underline{t} \in V$  and  $C$  is a  $\underline{t}$ -contour, then the radial function  $r_C$  satisfies the extra conditions

$$r_C(\alpha_i^+) > |t_i|, \quad i \in [0, 3],$$

where  $\alpha_i = \alpha_i(\underline{t})$  is defined by

$$(2.3.2) \quad \alpha_i^\pm := \frac{\arg(t_i^{\pm 1})}{2\pi}, \quad i \in [0, 3].$$

In particular, the unit circle  $T$  is a  $\underline{t}$ -contour if  $|t_i| < 1$  for all  $i \in [0, 3]$ . Observe furthermore that  $\alpha_i^\pm \neq 0, 1/2$  and that  $\alpha_i^- = 1 - \alpha_i^+$  for all  $i$ .

The convention will be used that a deformed circle  $C$  is counterclockwise oriented (i.e. has the orientation induced from the parametrization  $\phi_C$ ) when we integrate over  $C$ .

Let  $t \in (0, 1)$ ,  $\underline{t} \in V$  and let  $C$  be a deformed circle such that  $t_i q^j \notin C$  for  $i \in [0, 3]$  and  $j \in \mathbb{Z}_+$ . Let  $d\nu(z; \underline{t}; t)$  be the measure on  $C^n$  given by

$$(2.3.3) \quad d\nu(z; \underline{t}; t) := \Delta(z; \underline{t}; t) \frac{dz}{z}$$

for  $z \in C^n$  with  $\frac{dz}{z} := \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n}$  and with weight function  $\Delta(z; \underline{t}; t)$  given by

$$(2.3.4) \quad \Delta(z; \underline{t}; t) = \left( \prod_{j=1}^n w_c(z_j; \underline{t}) \right) \delta(z; t),$$

with  $w_c(x; \underline{t})$  (2.2.3) the weight function for the continuous part of the orthogonality measure of the Askey-Wilson polynomials, and  $\delta(z; t)$  given by

$$(2.3.5) \quad \delta(z; t) := \prod_{1 \leq i < j \leq n} (z_i z_j, z_i^{-1} z_j, z_i z_j^{-1}, z_i^{-1} z_j^{-1}; q)_\tau, \quad t = q^\tau.$$

The interaction factor  $\delta(z; t)$  is only present in the multivariable setting (i.e. when  $n > 1$ ). In particular the measure is independent of the deformation parameter  $t$  when  $n = 1$ , and coincides with the orthogonality measure of the Askey-Wilson polynomials as given in Theorem 2.2.1.

The measure  $d\nu(\cdot; \underline{t}; t)$  on  $C^n$  is well defined, since the poles of  $\Delta(\cdot; \underline{t}; t)$  do not intersect with the integration domain  $C^n$ . Indeed, the poles of the weight function  $\Delta(z; \underline{t}; t)$  lie on hyperplanes

$$(2.3.6) \quad z_i = t_m q^j \quad \text{or} \quad z_i = t_m^{-1} q^{-j}$$

with  $m \in [0, 3]$ ,  $j \in \mathbb{Z}_+$  and  $i \in [1, n]$  (the poles coming from  $w_c(z_i)$ ) and on hypersurfaces

$$(2.3.7) \quad z_k^{\varepsilon_k} z_l^{\varepsilon_l} = t^{-1} q^{-j}$$

with  $1 \leq k \neq l \leq n$ ,  $j \in \mathbb{Z}_+$  and  $\varepsilon_k, \varepsilon_l \in \{-1, 1\}$  (the poles coming from  $\delta(z; \underline{t})$ ). We have  $z \notin C^n$  for a pole  $z$  of the form (2.3.6) since  $C^{-1} = C$  and  $t_i q^j \notin C$  for  $i \in [0, 3]$  and  $j \in \mathbb{Z}_+$ . Similarly it follows from the definition of a deformed circle  $C$  that  $z \notin C^n$  for a pole  $z$  of the form (2.3.7).

In this section we will study the orthogonal polynomials related to the complex measure  $(C^n, d\nu(\cdot; \underline{t}; t))$  where  $C$  is an arbitrary  $\underline{t}$ -contour. We first show that the measure  $d\nu(\cdot; \underline{t}; t)$  is independent of the  $\underline{t}$ -contour  $C$  when integrating against  $\mathcal{W}$ -invariant Laurent polynomials. In order to obtain this result, we consider specific subsets of  $(\mathbb{C}^*)^n$  on which the interaction factor  $\delta(\cdot; t)$  is analytic.

Let  $C$  and  $\mathfrak{C}$  be deformed circles, with parametrization given by  $\phi_C(x) = r_C(x)e^{2\pi i x}$  respectively  $\phi_{\mathfrak{C}}(x) = r_{\mathfrak{C}}(x)e^{2\pi i x}$ . Let  $A^+(C, \mathfrak{C})$  be the open subset

$$(2.3.8) \quad A^+(C, \mathfrak{C}) := \{x \in [0, 1] \mid r_C(x) > r_{\mathfrak{C}}(x)\} \subset (0, 1).$$

Set

$$(2.3.9) \quad \Omega(C, \mathfrak{C}) := \Omega^+(C, \mathfrak{C}) \cup \Omega^+(\mathfrak{C}, C) \cup C,$$

where  $\Omega^+(C, \mathfrak{C})$  is given by

$$(2.3.10) \quad \Omega^+(C, \mathfrak{C}) := \bigcup_{x \in A^+(C, \mathfrak{C})} \{y(x)e^{2\pi i x} \mid r_C(x) \geq y(x) \geq r_{\mathfrak{C}}(x)\}.$$

The following properties of  $\Omega(C, \mathfrak{C}) \subset \mathbb{C}^*$  follow easily from the definitions:

(i)  $\Omega(C, \mathfrak{C}) = \Omega(\mathfrak{C}, C)$ .

We will use, in view of (i), the notation  $\Omega = \Omega(C, \mathfrak{C})$  when it is clear from the context which pair of contours  $C, \mathfrak{C}$  is meant.

(ii)  $\Omega^{-1} = \Omega$ .

(iii) The contour  $C$  can be deformed homotopically to  $\mathfrak{C}$  within  $\Omega$ .

We call  $\Omega \subset \mathbb{C}^*$  *the domain associated with the pair  $(C, \mathfrak{C})$* . We write  $\mathcal{O}_{\mathcal{W}}(\Omega^n)$  for the ring of  $\mathcal{W}$ -invariant functions  $f$  which are analytic on  $\Omega^n$ . We have now the following crucial lemma.

**LEMMA 2.3.3.** *Let  $t \in (0, 1)$  and let  $C, \mathfrak{C}$  be deformed circles satisfying the condition  $t(r_C(x)) < r_{\mathfrak{C}}(x)$  for all  $x \in A^+(C, \mathfrak{C})$ . Then  $\delta(\cdot; t) \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ .*

**PROOF.** Let  $C, \mathfrak{C}$  be deformed circles such that  $t(r_C(x)) < r_{\mathfrak{C}}(x)$  for  $x \in A^+(C, \mathfrak{C})$ . Let  $z \in (\mathbb{C}^*)^n$  such that  $z_k^{\varepsilon_k} z_l^{\varepsilon_l} = t^{-1} q^{-j}$  for some  $j \in \mathbb{Z}_+$ , some  $k \neq l$  and some  $\varepsilon_k, \varepsilon_l \in \{\pm 1\}$ . Write  $\beta_k := \arg(z_k)/2\pi$  and  $\beta_l := \arg(z_l)/2\pi$ . For the proof of the lemma it suffices to show that either  $z_k \notin \Omega$  or  $z_l \notin \Omega$ .

As an example, we check that either  $z_k \notin \Omega$  or  $z_l \notin \Omega$  when  $\beta_k \in A^+(C, \mathfrak{C})$  and  $z_k z_l = t^{-1} q^{-j}$  for some  $j \in \mathbb{Z}_+$ . Then  $\beta_k = 1 - \beta_l$  and  $\beta_l \in A^+(\mathfrak{C}, C)$ , so in particular  $r_C(\beta_k) = r_C(\beta_l)^{-1}$ ,  $r_{\mathfrak{C}}(\beta_k) = r_{\mathfrak{C}}(\beta_l)^{-1}$ . Suppose that  $z_l \in \Omega$ , then

$$|z_k| = t^{-1} q^{-j} |z_l^{-1}| \geq q^{-j} t^{-1} r_{\mathfrak{C}}(\beta_k) > q^{-j} r_C(\beta_k) \geq r_C(\beta_k),$$

hence  $z_k \notin \Omega$ . All the other cases are checked similarly.  $\square$

LEMMA 2.3.4. *Let  $\underline{t} \in V$ ,  $t \in (0, 1)$  and  $f \in A^{\mathcal{W}}$ . Then*

$$(2.3.11) \quad \iint_{z \in C^n} f(z) d\nu(z; \underline{t}; t)$$

*is independent of the choice of  $\underline{t}$ -contour  $C$ .*

PROOF. With the shorthand notation  $N_f(C)$  for the integral (2.3.11), we have to show that  $N_f(C) = N_f(\mathfrak{C})$  for arbitrary pairs of  $\underline{t}$ -contours  $(C, \mathfrak{C})$ .

Let  $\mathcal{L}$  be the collection of pairs of  $\underline{t}$ -contours  $(C, \mathfrak{C})$  for which  $A^+(C, \mathfrak{C})$  is a finite disjoint union of open intervals and for which  $t(r_C(x)) < r_{\mathfrak{C}}(x)$  for all  $x \in A^+(C, \mathfrak{C})$ . Fix a pair  $(C, \mathfrak{C}) \in \mathcal{L}$  and let  $\Omega$  be the associated domain. Since the four parameters  $t_0, t_1, t_2, t_3$  are in the interior of  $C$  and  $\mathfrak{C}$  we have  $w_c(\cdot; \underline{t}) \in \mathcal{O}_{\{\pm 1\}}(\Omega)$ , and by Lemma 2.3.3 we have  $\delta(\cdot; t) \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ . So Cauchy's Theorem implies that  $N_f(C) = N_f(\mathfrak{C})$ .

Suppose now that  $(C, \mathfrak{C})$  is an arbitrary pair of  $\underline{t}$ -contours. Then there exists a finite sequence of  $\underline{t}$ -contours  $C_0, C_1, \dots, C_s$  such that  $C_0 = \mathfrak{C}$ ,  $C_s = C$  and such that  $(C_i, C_{i-1}) \in \mathcal{L}$  for all  $i \in [1, s]$ . It follows that  $N_f(C) = N_f(\mathfrak{C})$ .  $\square$

Define for parameters  $\underline{t} \in V$  and  $t \in (0, 1)$  a symmetric bilinear form  $(\cdot, \cdot)_{\underline{t}, t}$  on  $A^{\mathcal{W}}$  by

$$(2.3.12) \quad (f, g)_{\underline{t}, t} := \frac{1}{(2\pi i)^n} \iint_{z \in C^n} f(z)g(z) d\nu(z; \underline{t}; t), \quad f, g \in A^{\mathcal{W}}$$

where  $C$  is a  $\underline{t}$ -contour. The bilinear form (2.3.12) is independent of the choice of  $\underline{t}$ -contour  $C$  by Lemma 2.3.4. An important tool for studying orthogonal polynomials with respect to the bilinear form  $(\cdot, \cdot)_{\underline{t}, t}$  is an explicit second order  $q$ -difference operator  $D = D_{\underline{t}, t}$  which preserves the algebra  $A^{\mathcal{W}}$  and which is symmetric with respect to  $(\cdot, \cdot)_{\underline{t}, t}$ . The second order  $q$ -difference operator  $D$  was introduced by Koornwinder [65]. It is explicitly given by

$$(2.3.13) \quad D := \sum_{j=1}^n (\phi_j^+(z)(T_j^+ - \text{Id}) + \phi_j^-(z)(T_j^- - \text{Id})),$$

where  $T_j^{\pm}$  is the  $q^{\pm 1}$ -shift in the  $j$ th coordinate,

$$(T_j^{\pm} f)(z) := f(z_1, \dots, z_{j-1}, q^{\pm 1} z_j, z_{j+1}, \dots, z_n),$$

and the functions  $\phi_j^+(z; \underline{t}; t)$  and  $\phi_j^-(z; \underline{t}; t)$  are given by

$$\phi_j^+(z; \underline{t}; t) := \frac{\prod_{i=0}^3 (1 - t_i z_j)}{(1 - z_j^2)(1 - qz_j^2)} \prod_{l \neq j} \frac{(1 - tz_l z_j)(1 - tz_l^{-1} z_j)}{(1 - z_l z_j)(1 - z_l^{-1} z_j)},$$

$$\phi_j^-(z; \underline{t}; t) := \phi_j^+(z^{-1}; \underline{t}; t),$$



where  $z^{-1} := (z_1^{-1}, \dots, z_n^{-1})$ . The  $BC$  type dominance order on  $\Lambda$  is defined by

$$(2.3.14) \quad \mu \leq \lambda \Leftrightarrow \sum_{j=1}^i \mu_j \leq \sum_{j=1}^i \lambda_j \quad (i \in [1, n])$$

for  $\lambda, \mu \in \Lambda$ .

REMARK 2.3.5. Choose for the root system  $R = R^+ \cup (-R^+)$  of type  $BC_n$ , the positive roots  $R^+$  by

$$R^+ = \{e_i\}_{i=1}^n \cup \{e_i \pm e_j\}_{1 \leq i < j \leq n} \cup \{2e_i\}_{i=1}^n,$$

with  $\{e_i\}_{i=1}^n$  the standard orthonormal basis for  $\mathbb{R}^n$ , then  $\Lambda$  coincides with the set of dominant weights, and  $\lambda > \mu$  for  $\lambda, \mu \in \Lambda$  iff  $\lambda - \mu$  is a sum of positive roots.

Koornwinder proved the following triangularity property of  $D$  (see [65, Lemma 5.2] and the remark after [121, Proposition 4.1]).

PROPOSITION 2.3.6. *Let  $\lambda \in \Lambda$ . For arbitrary  $\underline{t} \in \mathbb{C}^4$  and  $t \in \mathbb{C}$  we have*

$$(2.3.15) \quad Dm_\lambda = \sum_{\mu \leq \lambda} E_{\lambda, \mu} m_\mu$$

with  $E_{\lambda, \mu} = E_{\lambda, \mu}(\underline{t}; t) \in \mathbb{C}$  depending polynomially on the parameters  $\underline{t}$  and  $t$ . The leading term  $E_{\lambda, \lambda}(\underline{t}; t)$  will be denoted by  $E_\lambda(\underline{t}; t)$  and is given by

$$(2.3.16) \quad E_\lambda(\underline{t}; t) := \sum_{j=1}^n (q^{-1}t_0t_1t_2t_3t^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1)).$$

In particular,  $D$  preserves the algebra  $A^{\mathcal{W}}$ . An other important property of  $D$  is the symmetry of  $D$  with respect to the bilinear form  $(\cdot, \cdot)$ , i.e.

$$(2.3.17) \quad (D_{\underline{t}, t} f, g)_{\underline{t}, t} = (f, D_{\underline{t}, t} g)_{\underline{t}, t}, \quad f, g \in A^{\mathcal{W}}$$

for parameters  $\underline{t} \in V$  and  $t \in (0, 1)$ . Koornwinder [65, Lemma 5.3] proved (2.3.17) for parameters  $\underline{t}$  with  $|t_i| < 1$  (then the unit circle  $T$  can be chosen as  $\underline{t}$ -contour). By Proposition 2.3.6, (2.3.17) follows for  $\underline{t} \in V$  by analytic continuation.

Define now explicit expressions  $\mathcal{N}(\lambda; \underline{t}; t)$  for  $\lambda \in \Lambda$  by

$$(2.3.18) \quad \mathcal{N}(\lambda; \underline{t}; t) := 2^n n! \mathcal{N}^+(\lambda; \underline{t}; t) \mathcal{N}^-(\lambda; \underline{t}; t)$$

where  $\mathcal{N}^+(\lambda) := \mathcal{N}^+(\lambda; \underline{t}; t)$  is given by

$$(2.3.19) \quad \mathcal{N}^+(\lambda) := \prod_{i=1}^n \frac{(q^{2\lambda_i - 1} t^{2(n-i)} t_0 t_1 t_2 t_3; q)_\infty}{(q^{\lambda_i - 1} t^{n-i} t_0 t_1 t_2 t_3, q^{\lambda_i} t^{n-i} t_0 t_1, q^{\lambda_i} t^{n-i} t_0 t_2, q^{\lambda_i} t^{n-i} t_0 t_3; q)_\infty} \cdot \prod_{1 \leq j < k \leq n} \frac{(q^{\lambda_j + \lambda_k - 1} t^{2n-j-k} t_0 t_1 t_2 t_3, q^{\lambda_j - \lambda_k} t^{k-j}; q)_\infty}{(q^{\lambda_j + \lambda_k - 1} t^{2n-j-k+1} t_0 t_1 t_2 t_3, q^{\lambda_j - \lambda_k} t^{k-j+1}; q)_\infty},$$

and  $\mathcal{N}^-(\lambda) := \mathcal{N}^-(\lambda; \underline{t}; t)$  is given by

$$(2.3.20) \quad \mathcal{N}^-(\lambda) := \prod_{i=1}^n \frac{(q^{2\lambda_i} t^{2(n-i)} t_0 t_1 t_2 t_3; q)_\infty}{(q^{\lambda_i+1} t^{n-i}, q^{\lambda_i} t^{n-i} t_1 t_2, q^{\lambda_i} t^{n-i} t_1 t_3, q^{\lambda_i} t^{n-i} t_2 t_3; q)_\infty} \cdot \prod_{1 \leq j < k \leq n} \frac{(q^{\lambda_j + \lambda_k} t^{2n-j-k} t_0 t_1 t_2 t_3, q^{\lambda_j - \lambda_k + 1} t^{k-j}; q)_\infty}{(q^{\lambda_j + \lambda_k} t^{2n-j-k-1} t_0 t_1 t_2 t_3, q^{\lambda_j - \lambda_k + 1} t^{k-j-1}; q)_\infty}.$$

The following theorem extends the results of Koornwinder [65] (the orthogonality relations for the Koornwinder polynomials) and van Diejen in [17], Sahi [106] (the quadratic norm evaluations for the Koornwinder polynomials) to parameters  $\underline{t} \in V$  (in [65], [17] and [106] these results were obtained for a parameter domain such that  $|t_i| \leq 1$  for all  $i$ ). See also [120].

**THEOREM 2.3.7.** *For parameters  $(\underline{t}, t) \in V \times (0, 1)$  there exists a unique linear basis  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \in \Lambda}$  of  $A^{\mathcal{W}}$  such that*

$$(2.3.21) \quad P_\lambda(\cdot; \underline{t}; t) = m_\lambda + \sum_{\mu < \lambda} c_{\lambda, \mu}(\underline{t}, t) m_\mu, \quad c_{\lambda, \mu}(\underline{t}, t) \in \mathbb{C}$$

$$(2.3.22) \quad (P_\lambda(\cdot; \underline{t}; t), P_\mu(\cdot; \underline{t}; t))_{\underline{t}, t} = 0 \quad \text{if } \mu \neq \lambda.$$

Furthermore,  $P_\lambda(\cdot; \underline{t}; t)$  is an eigenfunction of  $D_{\underline{t}, t}$  with eigenvalue  $E_\lambda(\underline{t}; t)$  and we have the explicit evaluation formula

$$(2.3.23) \quad (P_\lambda(\cdot; \underline{t}; t), P_\lambda(\cdot; \underline{t}; t))_{\underline{t}, t} = \mathcal{N}(\lambda; \underline{t}; t)$$

for the quadratic norms of the polynomials  $P_\lambda$ .

**DEFINITION 2.3.8.** *The  $\mathcal{W}$ -invariant Laurent polynomial  $P_\lambda(\cdot; \underline{t}; t)$  is called the monic Koornwinder polynomial of degree  $\lambda \in \Lambda$ .*

We end this section with a sketch of the proof for Theorem 2.3.7 using the techniques of Koornwinder [65] and van Diejen [17]. For more details, we refer the reader to these two papers.

Fix arbitrary  $0 \neq \nu \in \Lambda$ . It is sufficient to prove the existence and uniqueness of a set of  $\mathcal{W}$ -invariant Laurent polynomials  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  satisfying (2.3.21) and (2.3.22) for  $\lambda, \mu \leq \nu$  and to prove the remaining assertions of the theorem for the polynomials  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$ .

We first define the polynomials  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  for a dense parameter domain  $U_\nu \subset V \times (0, 1)$ . The subset  $U_\nu$  is by definition the set of parameters  $(\underline{t}, t) \in V \times (0, 1)$  such that  $E_\mu(\underline{t}; t) \neq E_\lambda(\underline{t}; t)$  for all  $\lambda, \mu \leq \nu$ ,  $\lambda \neq \mu$ . Observe that  $U_\nu \subset V \times (0, 1)$  is open and dense since the eigenvalues  $\{E_\lambda(\underline{t}; t)\}_{\lambda \in \Lambda}$  are mutually different as polynomials in the parameters  $\underline{t}, t$ .

The polynomials  $P_\lambda(\cdot; \underline{t}; t) \in A^{\mathcal{W}}$  for  $(\underline{t}, t) \in U_\nu$  and  $\lambda \leq \nu$  are defined by

$$(2.3.24) \quad P_\lambda(\cdot; \underline{t}; t) := \left( \prod_{\mu < \lambda} \frac{D_{\underline{t}, t} - E_\mu(\underline{t}; t)}{E_\lambda(\underline{t}; t) - E_\mu(\underline{t}; t)} \right) m_\lambda$$

(cf. [82], [121]). It is an easy consequence of the triangularity of  $D$  (cf. Proposition 2.3.6) and of the Cayley-Hamilton Theorem that  $P_\lambda(\cdot; \underline{t}; t)$  (2.3.24) is the unique function of the form (2.3.21) which is an eigenfunction of  $D_{\underline{t}, t}$  with eigenvalue  $E_\lambda(\underline{t}; t)$ . The polynomials  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  (2.3.24) satisfy the orthogonality relations (2.3.22) for parameter values  $(\underline{t}, t) \in U_\nu$  since  $D$  is symmetric with respect to  $(\cdot, \cdot)$  and  $E_\mu(\underline{t}; t) \neq E_{\mu'}(\underline{t}; t)$  for  $\mu, \mu' \leq \lambda$ ,  $\mu \neq \mu'$ .

Next we establish the quadratic norm evaluations (2.3.23) for  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  with  $(\underline{t}, t) \in U_\nu$ . In the special case  $\lambda = 0$ , (2.3.23) reduces to

$$(2.3.25) \quad \begin{aligned} (1, 1)_{\underline{t}, t} &= \mathcal{N}(0; \underline{t}; t) \\ &= 2^n n! \prod_{i=1}^n \frac{(t, t^{2n-i-1} t_0 t_1 t_2 t_3; q)_\infty}{(q, t^{n-i+1}; q)_\infty \prod_{0 \leq j < k \leq 3} (t^{n-i} t_j t_k; q)_\infty} \end{aligned}$$

which was proved by Gustafson [31] for parameters  $t \in (0, 1)$  and  $\underline{t} \in \mathbb{C}^4$  with  $|t_i| < 1$  (since then the torus  $T$  can be chosen as  $\underline{t}$ -contour). The second equality follows by a straightforward computation (see also [83]). By analytic continuation, (2.3.25) is valid for parameters  $t \in (0, 1)$  and  $\underline{t} \in V$ .

For general  $\lambda$ , van Diejen [17] proved explicit Pieri formulas for the renormalized Koornwinder polynomials

$$(2.3.26) \quad p_\lambda(\cdot; \underline{t}; t) := c(\lambda; \underline{t}; t) P_\lambda(\cdot; \underline{t}; t), \quad c(\lambda; \underline{t}; t) := \frac{\mathcal{N}^+(0)}{\mathcal{N}^+(\lambda)} \prod_{j=1}^n (t_0 t^{n-j})^{\lambda_j}.$$

The renormalization constant  $c(\lambda; \underline{t}; t)$  is a rational expression in the parameters  $\underline{t}, t$ . The Pieri formulas give explicit expressions for the coefficients  $d_\lambda^{(r)}(\mu; \underline{t}; t)$  in the expansions

$$(2.3.27) \quad E_r(z; \underline{t}; t) p_\lambda(z; \underline{t}; t) = \sum_{\mu \leq \lambda + (1^n)} d_\lambda^{(r)}(\mu; \underline{t}; t) p_\mu(z; \underline{t}; t), \quad (r \in [1, n])$$

where  $\{E_r(z; \underline{t}; t)\}_{r=1}^n$  are explicit algebraic generators of the algebra of  $\mathcal{W}$ -invariant Laurent polynomials  $A^{\mathcal{W}}$  and where  $(1^n) := (1, \dots, 1) \in \Lambda$  is the  $n$ th fundamental weight (see [17] for the explicit formulas, or [20, Appendix B] where the notations are closer to the ones used in this Chapter).

In [17] the proof of the Pieri formulas was derived for a four parameter subfamily [17]. It turns out that Cherednik's affine Hecke-algebraic approach to the study of basic hypergeometric orthogonal polynomials related to root systems (cf. [12], [13], [14]) can in fact be worked out for the complete five parameter family of Koornwinder polynomials. This was worked out in detail by Noumi [90], Sahi [106], see also [120]. The results of Sahi [106] imply that van Diejen's Pieri formulas are valid for the full five parameter family of Koornwinder polynomials. In particular, (2.3.27) may be viewed as an identity

in the algebra of  $\mathcal{W}$ -invariant Laurent polynomials over the quotient field  $\mathbb{C}(\underline{t}, t)$ , where  $\underline{t}, t$  are considered as indeterminates.

The Pieri formulas and the orthogonality relations for the renormalized Koornwinder polynomials with real parameters  $t_i$  and  $|t_i| < 1$  allowed van Diejen (cf. [17, Theorem 4]) to reduce the norm computation for arbitrary  $\lambda$  to the case  $\lambda = 0 \in \Lambda$ . Gustafson's evaluation (2.3.25) then completes the evaluation for general  $\lambda$ . By taking a dense subset of the parameter domain  $U_\nu$  if necessary, exactly the same reduction can be done for the norm evaluations of the polynomials  $\{p_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  for parameters  $(\underline{t}, t) \in U_\nu$ . The extension of Gustafson's result (2.3.25) then completes the proof of (2.3.23) for  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  and  $(\underline{t}, t) \in U_\nu$ .

The polynomials  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  with parameter values  $(\underline{t}, t) \in U_\nu$  are uniquely characterized by (2.3.21) and (2.3.22) for  $\lambda, \mu \leq \nu$ , since their quadratic norms (2.3.23) are non-zero. Indeed, the functions

$$(2.3.28) \quad (\underline{t}, t) \mapsto \mathcal{N}^\pm(\lambda; \underline{t}; t) : V \times (0, 1) \rightarrow \mathbb{C}$$

are well defined, continuous functions which do not have zeros on the domain  $V \times (0, 1)$ . This is immediately clear except for  $\mathcal{N}^+(\lambda)$  with  $\lambda \in \Lambda$  and  $\lambda_n = 0$ . But then the expression for  $\mathcal{N}^+(\lambda)$  can be simplified, similarly as the simplification of the expression for  $\mathcal{N}(0)$  in (2.3.25), from which it follows that  $\mathcal{N}^+(\lambda; \underline{t}; t)$  is a well defined, continuous function of  $(\underline{t}, t) \in V \times (0, 1)$  without zeros.

The proof of the theorem can now be finished by extending these results to parameter values  $(\underline{t}, t) \in V \times (0, 1)$  using a continuity argument, as follows. The Koornwinder polynomial  $P_\lambda$  satisfies the following Gram-Schmidt formula,

$$(2.3.29) \quad P_\lambda(z; \underline{t}; t) = m_\lambda(z) - \sum_{\mu < \lambda} \frac{(m_\lambda, P_\mu(\cdot; \underline{t}; t))_{\underline{t}, t}}{\mathcal{N}(\mu; \underline{t}; t)} P_\mu(z; \underline{t}; t)$$

for  $(\underline{t}, t) \in U_\nu$  and  $\lambda \leq \nu$ . By induction, it follows from (2.3.29) that the coefficients  $c_{\lambda, \mu} : U_\nu \rightarrow \mathbb{C}$  in (2.3.21) uniquely extend to continuous functions  $c_{\lambda, \mu} : V \times (0, 1) \rightarrow \mathbb{C}$  for all  $\mu < \lambda \leq \nu$ . Hence existence and uniqueness of  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \leq \nu}$  as well as the other assertions follow now by continuity for all  $(\underline{t}, t) \in V \times (0, 1)$ . This completes the proof of the theorem.

**REMARK 2.3.9.** For  $n = 1$  the polynomials  $\{P_\lambda(z; \underline{t}) \mid \lambda \in \mathbb{Z}_+\}$  are independent of  $t$  and are the monic Askey-Wilson polynomials (2.2.2) as defined in Section 2.2. Theorem 2.3.7 reduces to the orthogonality relation and quadratic norm evaluation stated in Theorem 2.2.1. The renormalized Askey-Wilson polynomial  $p_\lambda(z; \underline{t})$  (2.3.26) is then exactly the  ${}_4\phi_3$  part of (2.2.1).

**REMARK 2.3.10.** The renormalization constant  $c(\lambda; \underline{t}; t)$  (2.3.26) is easily seen to be regular and non-zero at  $(\underline{t}, t) \in V \times (0, 1)$  for all  $\lambda \in \Lambda$ . Hence the renormalized Koornwinder polynomials  $\{p_\lambda(z; \underline{t}; t)\}_{\lambda \in \Lambda}$  form an orthogonal basis of  $A^{\mathcal{W}}$  with respect to the bilinear form  $(\cdot, \cdot)_{\underline{t}, t}$  for all parameter values  $(\underline{t}, t) \in V \times (0, 1)$ .

REMARK 2.3.11. Several elementary properties of the Koornwinder polynomials can be deduced using the fact that for generic parameters,  $P_\lambda(\cdot; \underline{t}; t)$  is the unique function of the form (2.3.21) which is an eigenfunction of  $D_{\underline{t}, t}$  with eigenvalue  $E_\lambda(\underline{t}; t)$ . For instance, it follows that the Koornwinder polynomial  $P_\lambda(\cdot; \underline{t}; t)$  is symmetric in the four parameters  $\underline{t}$ , and that  $P_\lambda(z; -\underline{t}; t) = (-1)^{|\lambda|} P_\lambda(-z; \underline{t}; t)$  where  $|\lambda| := \sum_{i=1}^n \lambda_i$ ,  $-z := (-z_1, \dots, -z_n)$  and, similarly,  $-\underline{t} = (-t_0, -t_1, -t_2, -t_3)$ .

#### 2.4. Residue calculus for the orthogonality measure $d\nu$

In this section a residue calculus is developed for integrals of the form

$$(2.4.1) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^n} f(z) d\nu(z) = \frac{1}{(2\pi i)^n} \iint_{z \in C^n} f(z) \Delta(z) \frac{dz}{z}, \quad f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$$

when  $C^n$  is shifted to  $\mathfrak{C}^n$ , where  $\Omega \subset \mathbb{C}^*$  is the domain associated with the pair  $(C, \mathfrak{C})$  and  $(C, \mathfrak{C})$  is a so-called  $(n, t_0)$ -residue pair, which is defined as follows.

DEFINITION 2.4.1. Let  $\underline{t} = (t_0, t_1, t_2, t_3) \in V$  ( $V$  as given in Definition 2.3.1). A pair of contours  $(C, \mathfrak{C})$  is called a  $(n, t_0)$ -residue pair if  $C$  and  $\mathfrak{C}$  are deformed circles satisfying the following three properties.

- (i) The subset  $A^+(C, \mathfrak{C})$  (2.3.8) is an open interval for which  $\alpha_0^+ \in A^+(C, \mathfrak{C})$  but  $\alpha_i^\pm \notin A^+(C, \mathfrak{C})$  ( $i = 1, 2, 3$ );
- (ii)  $t_i q^r \notin C \cup \mathfrak{C}$  for  $r \in \mathbb{Z}_+$  and  $i \in [0, 3]$ ;
- (iii)  $t_0 t^p q^r \notin \mathfrak{C}$  for  $p \in [-1, n-1]$  and  $r \in \mathbb{Z}$ .

The poles which are picked up when deforming  $C^n$  to  $\mathfrak{C}^n$  in (2.4.1) for a  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$  will only depend on  $q, t$  and  $t_0$ . We therefore fix in this section  $t \in (0, 1)$  and  $t_1, t_2, t_3 \in \mathbb{C}^*$  such that  $\#\{\alpha_i^+, \alpha_i^- \mid i = 1, 2, 3\} = 6$  ( $\alpha_i^\pm$  given by (2.3.2)) and simplify the notations by omitting the dependence on these parameters. For instance, we will write  $w_c(x; t_0)$  instead of  $w_c(x; \underline{t})$ ,  $w_d(x; t_0)$  instead of  $w_d(x; t_0; t_1, t_2, t_3)$ , etc.

Observe that due to the symmetry of  $\Delta(z; \underline{t}; t)$  in the four parameters  $t_0, t_1, t_2$  and  $t_3$ , all the results on the residue calculus for  $(n, t_0)$ -residue pairs can be reformulated for  $(n, t_i)$ -residue pairs with  $i \in [0, 3]$  arbitrary by relabeling the parameters  $\underline{t}$ .

The measure which is obtained after deforming the contour  $C^n$  in (2.4.1) to  $\mathfrak{C}^n$  for a  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$  is at first site rather complicated. We will therefore first give the result for the special case  $t = q^k$  ( $k \in \mathbb{N}$ ), in which case the answer as well as the proof is much simpler. The rather dramatic simplification of the proof is a consequence of the fact that the interaction factor  $\delta(\cdot; q^k)$  is analytic on  $(\mathbb{C}^*)^n$ , which is certainly not the case for general  $t$ . Despite this simplification, the answer has the same form as in the general case.

LEMMA 2.4.2. Let  $\underline{t} \in V$ ,  $t = q^k$  with  $k \in \mathbb{N}$  and let  $(C, \mathfrak{C})$  be an  $(n, t_0)$ -residue pair. Let  $M$  be the smallest positive integer such that  $|t_0 q^M| < r_C(\alpha_0^+)$  and let  $N$  be the

largest positive integer such that  $|t_0 q^N| > r c(\alpha_0^+)$ . Then

$$(2.4.2) \quad \frac{1}{(2\pi i)^n} \iint_{z \in \mathbb{C}^n} f(z) d\nu(z) = \sum_{r=0}^n \frac{2^r \binom{n}{r}}{(2\pi i)^{n-r}} \sum_{\substack{z_i \in \{t_0 q^l\}_{l=1, \dots, r}^N \\ (z_{r+1}, \dots, z_n) \in \mathbb{C}^{n-r}}} \iint f(z) \cdot \delta(z; q^k) \prod_{i=1}^r w_d(z_i; t_0) \prod_{j=r+1}^n w_c(z_j) \frac{dz_j}{z_j}$$

for  $f \in A^{\mathcal{W}}$ .

PROOF. Write  $I_n(f)$  for the left hand side of (2.4.2) and  $\tilde{I}_n(f)$  for the right hand side of (2.4.2). We prove  $I_n(f) = \tilde{I}_n(f)$  for  $f \in A^{\mathcal{W}}$  by induction on the number of variables  $n$ . The equality  $I_1(f) = \tilde{I}_1(f)$  for  $f \in \mathbb{C}[z + z^{-1}]$  is well-known (see [7], or Section 2.2). For  $n > 1$ , we obtain from Cauchy's Theorem and the fact that  $\delta(\cdot; q^k) \in \mathcal{O}_{\mathcal{W}}((\mathbb{C}^*)^n)$ ,

$$(2.4.3) \quad I_n(f) = \frac{1}{2\pi i} \int_{z_1 \in \mathbb{C}} w_c(z_1) I_{n-1}(f_{z_1}) \frac{dz_1}{z_1} + 2 \sum_{z_1 = \{t_0 q^l\}_{l=1}^N} w_d(z_1; t_0) I_{n-1}(f_{z_1})$$

for  $f \in A^{\mathcal{W}}$ , where  $f_x(w)$  ( $x \in \mathbb{C}^*$ ) is the  $\mathcal{W}$ -invariant Laurent polynomial in  $n-1$  variables  $w = (w_1, \dots, w_{n-1})$  given by

$$f_x(w) := f(x, w) \prod_{i=1}^{n-1} (xw_i, xw_i^{-1}, x^{-1}w_i, x^{-1}w_i^{-1}; q)_k.$$

By the induction hypotheses we may replace  $I_{n-1}(f_x)$  by  $\tilde{I}_{n-1}(f_x)$  in (2.4.3). Then the equality  $I_n(f) = \tilde{I}_n(f)$  follows from the  $\mathcal{W}$ -invariance of  $f$  and  $\delta(\cdot; q^k)$ , and from the formula

$$\frac{1}{2\pi i} \frac{2^r \binom{n-1}{r}}{(2\pi i)^{n-1-r}} + 2 \frac{2^{r-1} \binom{n-1}{r-1}}{(2\pi i)^{n-r}} = \frac{2^r \binom{n}{r}}{(2\pi i)^{n-r}} \quad (r \in [0, n]),$$

where we use the convention that  $\binom{m}{r} = 0$  if  $r < 0$  and if  $r > m$ .  $\square$

Since  $\delta(z; q^k) = 0$  if  $z_i = q^l z_j$  for some  $i \neq j$  and some  $l \in [0, k-1]$  and using  $\mathfrak{S}$ -symmetry properties of the weights in the right hand side of (2.4.2), it follows that the discrete parts are actually supported on points of the form

$$(t_0 q^{\lambda_1}, t_0 t q^{\lambda_2}, \dots, t_0 t^{r-1} q^{\lambda_r})$$

with  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r$  positive integers and  $t = q^k$ . This turn out to be also the case for arbitrary  $t \in (0, 1)$ , as will be shown in a moment. We first need to introduce some notations. Set

$$(2.4.4) \quad P(r) := \{\lambda \in \mathbb{Z}_+^r \mid \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_r\}$$

and set  $\lambda_0 := 0$  for arbitrary  $\lambda \in P(r)$ . We write  $\rho_i = \rho_i(t_0; t) := t_0 t^{i-1}$  for  $i \in \mathbb{Z}$  and, for  $\lambda \in P(r)$ ,

$$(2.4.5) \quad \rho q^\lambda := (\rho_1 q^{\lambda_1}, \rho_2 q^{\lambda_2}, \dots, \rho_r q^{\lambda_r}) = (t_0 q^{\lambda_1}, t_0 t q^{\lambda_2}, \dots, t_0 t^{r-1} q^{\lambda_r}).$$

Define  $D(r) = D(r; C, \mathfrak{C}; t_0)$  for  $r \in [1, n]$  by

$$(2.4.6) \quad D(r) := \{\rho q^\lambda \mid \lambda \in P(r) \text{ and } r_C(\alpha_0^+) > |\rho_i q^{\lambda_i}| > r_{\mathfrak{C}}(\alpha_0^+) \quad (i \in [1, r])\}.$$

Observe that for  $\omega \in D(r)$  we have  $\omega_i \in \text{int}(\Omega)$  for all  $i$ , where  $\text{int}(\Omega)$  is the interior of  $\Omega$ . For  $\rho q^\lambda \in D(r)$ , we set

$$(2.4.7) \quad \Delta^{(d)}(\rho q^\lambda; t_0) := \left( \prod_{j=1}^r w_d(\rho_j q^{\lambda_j}; \rho_j q^{\lambda_{j-1}}) \right) \delta_d(\rho q^\lambda)$$

where  $w_d$  is given by (2.2.5), (2.2.7), and with interaction factor

$$(2.4.8) \quad \delta_d(\rho q^\lambda) := \prod_{1 \leq k < l \leq r} \frac{(\rho_k^{-1} \rho_l q^{\lambda_l - \lambda_k}, \rho_k^{-1} \rho_l^{-1} q^{-\lambda_k - \lambda_l}; q)_\tau}{(\rho_k \rho_l q^{\lambda_{k-1} + \lambda_l}, \rho_k \rho_l^{-1} q^{\lambda_{k-1} - \lambda_l}; q)_{\lambda_k - \lambda_{k-1}}} \quad (t = q^\tau).$$

Observe that the weight function  $w_c(x; \rho_j q^{\lambda_j})/x$  has a simple pole at  $x = \rho_j q^{\lambda_j}$  since  $\lambda_j - \lambda_{j-1} \in \mathbb{Z}_+$  and  $t \in V$ ; hence the factors  $w_d(\rho_j q^{\lambda_j}; \rho_j q^{\lambda_{j-1}})$  in (2.4.7) are non-zero. The discrete parts of the measure which will appear by deforming  $C^n$  to  $\mathfrak{C}^n$  in (2.4.1) will involve the weights  $\Delta^{(d)}$ . Observe furthermore that for  $r = 1$  and  $\rho q^\lambda = t_0 q^i \in D(1)$ , we have  $\Delta^{(d)}(t_0 q^i; t_0) = w_d(t_0 q^i; t_0)$ . For  $\rho q^\lambda \in D(r)$  and  $z \in \mathfrak{C}^{n-r}$ , set

$$(2.4.9) \quad d\nu_r(\rho q^\lambda, z; t_0) = \Delta_r(\rho q^\lambda, z; t_0) \frac{dz}{z}$$

with weight function  $\Delta_r(\rho q^\lambda, z; t_0)$  given by

$$(2.4.10) \quad \Delta_r(\rho q^\lambda, z; t_0) := \Delta^{(d)}(\rho q^\lambda; t_0) \Delta(z; t_0) \delta_c(\rho q^\lambda; z),$$

where  $\Delta(z; t_0)$  is the weight function (2.3.4) defined with respect to the variables  $z = (z_1, \dots, z_{n-r})$  and where  $\delta_c(\rho q^\lambda; z)$  is an interaction factor given by

$$(2.4.11) \quad \delta_c(\rho q^\lambda; z) := \prod_{\substack{1 \leq k \leq r \\ 1 \leq l \leq n-r}} (\rho_k q^{\lambda_k} z_l, \rho_k q^{\lambda_k} z_l^{-1}, \rho_k^{-1} q^{-\lambda_k} z_l, \rho_k^{-1} q^{-\lambda_k} z_l^{-1}; q)_\tau$$

with  $t = q^\tau$ . In particular,  $\Delta_n(\rho q^\lambda; t_0) = \Delta^{(d)}(\rho q^\lambda; t_0)$  for  $\rho q^\lambda \in D(n)$ . The measure  $d\nu_r(\rho q^\lambda, z; t_0)$  is well defined on  $D(r) \times \mathfrak{C}^{n-r}$  since the denominator of  $\Delta_r(\rho q^\lambda, z; t_0)$  is non-zero by properties (ii) and (iii) of the  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$  (Definition 2.4.1). We call  $d\nu_r$  the  $r$ th measure associated with the  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$ .

PROPOSITION 2.4.3. *Let  $(C, \mathfrak{C})$  be a  $(n, t_0)$ -residue pair and let  $\Omega = \Omega(C, \mathfrak{C})$  be the associated domain. Let  $d\nu_r$  be the  $r$ th measure associated with  $(C, \mathfrak{C})$ , then*

$$(2.4.12) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^n} f(z) d\nu(z) = \frac{1}{(2\pi i)^n} \iint_{z \in \mathfrak{C}^n} f(z) d\nu(z) + \sum_{r=1}^n \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \sum_{\omega \in D(r)} \iint_{z \in \mathfrak{C}^{n-r}} f(\omega, z) d\nu_r(\omega, z)$$

for  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ , where  $(u)_r := \prod_{i=0}^{r-1} (u+i)$  is the shifted factorial.

REMARK 2.4.4. It is easy to show that Proposition 2.4.3 reduces to the statement in Lemma 2.4.2 for  $t = q^k$  ( $k \in \mathbb{N}$ ) using the fact that  $\delta(z; q^k) = 0$  if  $z_i = q^l z_j$  for some  $i \neq j$  and some  $l \in [0, k-1]$  and the fact that

$$w_d(x; e_i q^l) = (e_i x, e_i x^{-1}; q)_l w_d(x; e_i), \quad (x = e_i q^{l+m}, m \in \mathbb{Z}_+).$$

In the next lemma the proof of Proposition 2.4.3 is given for  $(n, t_0)$ -residue pairs  $(C, \mathfrak{C})$  such that the interaction factor  $\delta(\cdot; t)$  is analytic on  $(\Omega(C, \mathfrak{C}))^n$ .

LEMMA 2.4.5. *Suppose that  $(C, \mathfrak{C})$  is a  $(n, t_0)$ -residue pair such that*

$$(2.4.13) \quad t(r_C(x)) < r_{\mathfrak{C}}(x), \quad \forall x \in A^+(C, \mathfrak{C}).$$

Then (2.4.12) is valid.

PROOF. Fix a  $(n, t_0)$ -residue pairs  $(C, \mathfrak{C})$  satisfying (2.4.13). We will prove by induction on  $l \in [0, n]$  that

$$(2.4.14) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^l \times \mathfrak{C}^{n-l}} f(z) \Delta(z; t_0) \frac{dz}{z} = \frac{1}{(2\pi i)^n} \iint_{z \in \mathfrak{C}^n} f(z) \Delta(z; t_0) \frac{dz}{z} + \frac{2l}{(2\pi i)^{n-1}} \sum_{i \in I} \iint_{z \in \mathfrak{C}^{n-1}} f(t_0 q^i, z) \Delta_1(t_0 q^i, z; t_0) \frac{dz}{z}$$

for  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ , where

$$(2.4.15) \quad I = \{i \in \mathbb{Z}_+ \mid r_C(\alpha_0^+) > |t_0 q^i| > r_{\mathfrak{C}}(\alpha_0^+)\}.$$

Then (2.4.12) is the special case  $l = n$  in (2.4.14), since  $D(r) = \emptyset$  for  $r > 1$  by (2.4.13).

For  $l = 0$ , (2.4.14) is trivial. Let  $l \in [1, n]$ . Since  $\delta(\cdot; t) \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$  by Lemma 2.3.3, we can shift  $C$  to  $\mathfrak{C}$  for the first variable  $z_1$  in the left hand side of (2.4.14) and we obtain by (2.2.5), by the  $\mathcal{W}$ -invariance of  $\Delta(z)$  and by Cauchy's Theorem

$$(2.4.16) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^l \times \mathfrak{C}^{n-l}} f(z) \Delta(z; t_0) \frac{dz}{z} = \frac{1}{(2\pi i)^n} \iint_{z \in C^{l-1} \times \mathfrak{C}^{n-l+1}} f(z) \Delta(z; t_0) \frac{dz}{z} + \frac{2}{(2\pi i)^{n-1}} \sum_{i \in I} \iint_{z \in C^{l-1} \times \mathfrak{C}^{n-l}} f(t_0 q^i, z) \Delta_1(t_0 q^i, z; t_0) \frac{dz}{z}$$



for  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ . Here we have used that the residue at  $z_1 = t_0^{-1}q^{-i}$  ( $i \in I$ ) of the function  $\Delta(z_1, z'; t_0)/z_1$  is equal to  $-\Delta_1(t_0q^i, z'; t_0)$  in view of (2.2.6). The weight function  $\Delta_1(t_0q^i, z; t_0)$  in (2.4.16) can be rewritten as

$$(2.4.17) \quad \Delta_1(t_0q^i, z; t_0) = h(t_0q^i, z; t_0)\Delta(z; tt_0q^i)$$

with

$$(2.4.18) \quad h(t_0q^i, z; t_0) = w_d(t_0q^i; t_0) \prod_{s=1}^{n-1} \frac{(t_0^{-1}q^{-i}z_s, t_0^{-1}q^{-i}z_s^{-1}; q)_{\tau}}{(t_0z_s, t_0z_s^{-1}; q)_i} \quad (t = q^{\tau}).$$

Formula (2.4.17) follows by interchanging the factor  $(t_0z_s, t_0z_s^{-1}; q)_{\infty}$  in the denominator of the weight  $w_c(z_s; t_0)$  with the factor  $(tt_0q^iz_s, tt_0q^iz_s^{-1}; q)_{\infty}$  in the denominator of the interaction factor  $\delta_c(t_0q^i; z)$  for  $s \in [1, n-1]$ . We have  $\Delta(\cdot; tt_0q^i) \in \mathcal{O}_{\mathcal{W}}(\Omega^{n-1})$  for  $i \in I$  by (2.4.13) and Lemma 2.3.3. We claim that  $h(t_0q^i, \cdot; t_0) \in \mathcal{O}_{\mathcal{W}}(\Omega^{n-1})$  for  $i \in I$ . Indeed, it is sufficient to check that the map

$$(2.4.19) \quad x \mapsto \frac{(t_0^{-1}q^{-i}x, t_0^{-1}q^{-i}x^{-1}; q)_{\infty}}{(t_0x, t_0x^{-1}; q)_i (tt_0^{-1}q^{-i}x, tt_0^{-1}q^{-i}x^{-1}; q)_{\infty}}$$

is analytic on  $\Omega$  when  $i \in I$ . The zeros of the factor  $(t_0x, t_0x^{-1}; q)_i$  in the denominator are compensated by zeros in the numerator. Next, we check that  $(tt_0^{-1}q^{-i}x; q)_{\infty}$  is non-zero for  $x \in \Omega$  and  $i \in I$ . Now  $(tt_0^{-1}q^{-i}x; q)_{\infty} = 0$  iff  $x = t^{-1}q^{i-m}t_0$  for some  $m \in \mathbb{Z}_+$ . In particular, we must have  $\arg(x) = \arg(t_0) = 2\pi\alpha_0^+$ . Since  $i \in I$ , we have for  $m \in \mathbb{Z}_+$ ,

$$|t^{-1}q^{i-m}t_0| > t^{-1}r_{\mathfrak{c}}(\alpha_0^+)q^{-m} \geq t^{-1}r_{\mathfrak{c}}(\alpha_0^+) > r_C(\alpha_0^+),$$

where the last inequality is obtained from the extra condition (2.4.13). Since  $x \in \Omega$  with  $\arg(x) = 2\pi\alpha_0^+$  implies that  $r_{\mathfrak{c}}(\alpha_0^+) \leq |x| \leq r_C(\alpha_0^+)$ , it follows that  $(tt_0^{-1}q^{-i}x; q)_{\infty} \neq 0$  for  $x \in \Omega$  and  $i \in I$ . Since  $\Omega^{-1} = \Omega$ , we then also have  $(tt_0^{-1}q^{-i}x^{-1}; q)_{\infty} \neq 0$  for  $x \in \Omega$  and  $i \in I$ . Thus the map given by (2.4.19) is analytic on  $\Omega$  if  $i \in I$ . In particular,

$$f(t_0q^i, \cdot)\Delta_1(t_0q^i, \cdot; t_0) \in \mathcal{O}_{\mathcal{W}}(\Omega^{n-1})$$

for  $i \in I$ , so we obtain by Cauchy's Theorem and (2.4.16),

$$(2.4.20) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^l \times \mathfrak{C}^{n-l}} f(z)\Delta(z; t_0) \frac{dz}{z} = \frac{1}{(2\pi i)^n} \iint_{z \in C^{l-1} \times \mathfrak{C}^{n-l+1}} f(z)\Delta(z; t_0) \frac{dz}{z} \\ + \frac{2}{(2\pi i)^{n-1}} \sum_{i \in I} \iint_{z \in \mathfrak{C}^{n-1}} f(t_0q^i, z)\Delta_1(t_0q^i, z; t_0) \frac{dz}{z}$$

for  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ . Then (2.4.14) follows by applying the induction hypotheses on the integral over  $C^{l-1} \times \mathfrak{C}^{n-l+1}$  in (2.4.20).  $\square$

Lemma 2.4.5 can be used to prove Proposition 2.4.3 inductively. The following definition will be used to formulate the induction hypotheses.

DEFINITION 2.4.6. Let  $(C, \mathfrak{C})$  be a  $(n, t_0)$ -residue pair and let  $A^+(C, \mathfrak{C})$  (2.3.8) be the corresponding open interval. A sequence of closed contours  $(C_0, \dots, C_s)$  is called a  $(n, t_0)$ -resolution for  $(C, \mathfrak{C})$  if the contours  $C_l$  are deformed circles satisfying the following four conditions (we write  $r_l$  for the (radial) functions  $r_{C_l}$  in the parametrization  $\phi_{C_l}$  of  $C_l$ ):

- (i)  $C_0 = \mathfrak{C}$  and  $C_s = C$ ;
- (ii)  $r_l(x) = r_{\mathfrak{C}}(x) = r_C(x)$  for  $x \notin A^+(C, \mathfrak{C}) \cup A^+(\mathfrak{C}, C)$  and  $l \in [0, s]$ ;
- (iii)  $t(r_{l+1}(x)) < r_l(x) < r_{l+1}(x)$  for  $x \in A^+(C, \mathfrak{C})$  and  $l \in [0, s-1]$ ;
- (iv)  $t_0 t^p q^r \notin C_l$  for  $p \in [-1, n-1]$ ,  $r \in \mathbb{Z}$  and  $l \in [1, s-1]$ .

We call  $s$  the length of the resolution.

Observe that there exists a  $(n, t_0)$ -resolution for every  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$ . If  $(C_0, \dots, C_s)$  is a  $(n, t_0)$ -resolution for a  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$ , then  $(C_l, C_{l-1})$  is a  $(n, t_0)$ -residue pair satisfying the extra condition (2.4.13) used to prove Lemma 2.4.5 ( $l \in [1, s]$ ). Proposition 2.4.3 can now be proved by induction on the length of the resolution.

PROOF OF PROPOSITION 2.4.3. Suppose that for all  $n \in \mathbb{N}$  and all  $t_0 \in \mathbb{C}^*$  with  $\underline{t} = (t_0, t_1, t_2, t_3) \in V$ , Proposition 2.4.3 has been proved for  $(n, t_0)$ -residue pairs which have a  $(n, t_0)$ -resolution of length  $\leq s-1$ , where  $s \geq 2$ .

Fix arbitrary  $n \in \mathbb{N}$  and  $t_0 \in \mathbb{C}^*$  such that  $\underline{t} = (t_0, t_1, t_2, t_3) \in V$ . The induction step is clear for  $n = 1$ , so we may assume that  $n > 1$ . Let  $(C, \mathfrak{C})$  be a  $(n, t_0)$ -residue pair with a  $(n, t_0)$ -resolution  $(C_0, \dots, C_s)$  of length  $s$ . It suffices to prove (2.4.12) for the  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$ . We write  $\Omega^{(l)}$  and  $\Omega_{(l)}$  for the domains associated with the  $(n, t_0)$ -residue pairs  $(C_l, C_{l-1})$  and  $(C_l, \mathfrak{C})$  respectively ( $l \in [1, s]$ ). Note that  $\Omega_{(1)} \subset \Omega_{(2)} \subset \dots \subset \Omega_{(s)} = \Omega$  where  $\Omega$  is the domain associated with the  $(n, t_0)$ -residue pair  $(C, \mathfrak{C})$ . By (2.4.14) and (2.4.17), we have

$$(2.4.21) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^n} f(z) \Delta(z; t_0) \frac{dz}{z} = \frac{1}{(2\pi i)^n} \iint_{z \in (C_{s-1})^n} f(z) \Delta(z; t_0) \frac{dz}{z} + \frac{2n}{(2\pi i)^{n-1}} \sum_{i \in I_s} \iint_{z \in (C_{s-1})^{n-1}} f_i(z) \Delta(z; t_0 q^i) \frac{dz}{z}$$

for  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ , where

$$(2.4.22) \quad I_s := \{i \in \mathbb{Z}_+ \mid r_s(\alpha_0^+) > |t_0 q^i| > r_{s-1}(\alpha_0^+)\}$$

and  $f_i(z) := f(t_0 q^i, z) h(t_0 q^i, z; t_0)$  with  $h$  given by (2.4.18). We will apply the induction hypotheses on all the terms in the right hand side of (2.4.21). For the integral over  $(C_{s-1})^n$  note that  $(C_0, \dots, C_{s-1})$  is a  $(n, t_0)$ -resolution of length  $s-1$  for the

$(n, t_0)$ -residue pair  $(C_{s-1}, \mathfrak{C})$ . Hence, by the induction hypotheses,

$$(2.4.23) \quad \begin{aligned} & \frac{1}{(2\pi i)^n} \iint_{z \in (C_{s-1})^n} f(z) \Delta(z; t_0) \frac{dz}{z} = \frac{1}{(2\pi i)^n} \iint_{z \in \mathfrak{C}^n} f(z) \Delta(z; t_0) \frac{dz}{z} \\ & + \sum_{r=1}^n \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \sum_{\omega \in D(r; C_{s-1}, \mathfrak{C}; t_0)} \iint_{z \in \mathfrak{C}^{n-r}} f(\omega, z) \Delta_r(\omega, z; t_0) \frac{dz}{z} \end{aligned}$$

for all  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ .

Now fix an  $i \in I_s$ . In the proof of Lemma 2.4.5 it was shown that  $h(t_0 q^i, \cdot; t_0) \in \mathcal{O}_{\mathcal{W}}((\Omega^{(s)})^{n-1})$ . In fact it follows from the proof that  $h(t_0 q^i, \cdot; t_0) \in \mathcal{O}_{\mathcal{W}}(\Omega^{n-1})$ . In particular we have  $f_i \in \mathcal{O}_{\mathcal{W}}((\Omega_{(s-1)})^{n-1})$  for  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$ . Observe furthermore that  $(tt_0 q^i, t_1, t_2, t_3) \in V$  since  $\arg(tt_0 q^i) = \arg(t_0)$ , and that the sequence  $(C_0, \dots, C_{s-1})$  is a  $(n-1, tt_0 q^i)$ -resolution of length  $s-1$  for the  $(n-1, tt_0 q^i)$ -residue pair  $(C_{s-1}, \mathfrak{C})$ . So the induction hypotheses can be applied to all the terms in the second line of (2.4.21), and we obtain

$$(2.4.24) \quad \begin{aligned} & \frac{2n}{(2\pi i)^{n-1}} \iint_{z \in (C_{s-1})^{n-1}} f_i(z) \Delta(z; tt_0 q^i) \frac{dz}{z} = \frac{2n}{(2\pi i)^{n-1}} \iint_{z \in \mathfrak{C}^{n-1}} f_i(z) \Delta(z; tt_0 q^i) \frac{dz}{z} \\ & + \sum_{r=2}^n \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \sum_{\omega \in D(r-1; C_{s-1}, \mathfrak{C}; tt_0 q^i)} \iint_{z \in \mathfrak{C}^{n-r}} f_i(\omega, z) \Delta_{r-1}(\omega, z; tt_0 q^i) \frac{dz}{z} \end{aligned}$$

for  $f \in \mathcal{O}_{\mathcal{W}}(\Omega^n)$  and  $i \in I_s$ .

Substitution of (2.4.23) and (2.4.24) in the right hand side of (2.4.21) completes the proof of (2.4.12), since

$$\begin{aligned} D(1; C, \mathfrak{C}; t_0) &= D(1; C_{s-1}, \mathfrak{C}; t_0) \cup \{t_0 q^i\}_{i \in I_s}, \\ D(r; C, \mathfrak{C}; t_0) &= D(r; C_{s-1}, \mathfrak{C}; t_0) \cup \bigcup_{i \in I_s} \{(t_0 q^i, \omega) \mid \omega \in D(r-1; C_{s-1}, \mathfrak{C}; tt_0 q^i)\} \end{aligned}$$

disjoint unions ( $r \in [2, n]$ ) and

$$(2.4.25) \quad \begin{aligned} f_i(z) \Delta(z; tt_0 q^i) &= f(t_0 q^i, z) \Delta_1(t_0 q^i, z; t_0), \\ f_i(\omega, z) \Delta_{r-1}(\omega, z; tt_0 q^i) &= f(t_0 q^i, \omega, z) \Delta_r(t_0 q^i, \omega, z; t_0) \end{aligned}$$

for  $i \in I_s$ ,  $r \in [2, n]$  and  $\omega \in D(r-1; C_{s-1}, \mathfrak{C}; tt_0 q^i)$ .  $\square$

## 2.5. Multivariable $q$ -Racah polynomials

In this section the Koornwinder polynomials are studied for parameter values  $(\underline{t}, t)$  satisfying a particular truncation condition. By applying the residue calculus (cf. Section 2.4) to the results in Theorem 2.3.7 it will be shown that the Koornwinder polynomials for these parameter values are orthogonal with respect to a finite, discrete measure and the corresponding quadratic norms will be computed. The orthogonality relations which

will be obtained reduce to the orthogonality relations of the  $q$ -Racah polynomials in the one variable setting ( $n = 1$ ) (cf. Theorem 2.2.2).

For  $\lambda \in P(r)$  set

$$(2.5.1) \quad \Delta^{qR}(\rho q^\lambda; \underline{t}; t) := \prod_{i=1}^r \left( \frac{(q\rho_i^2; q)_{2\lambda_i}}{(\rho_i^2; q)_{2\lambda_i} (q^{-1}t_0 t_1 t_2 t_3 t^{2i-2})^{\lambda_i}} \prod_{j=0}^3 \frac{(t_j \rho_i; q)_{\lambda_i}}{(qt_j^{-1} \rho_i; q)_{\lambda_i}} \right) \\ \cdot \prod_{1 \leq k < l \leq r} \frac{(q\rho_k \rho_l, t\rho_k \rho_l; q)_{\lambda_k + \lambda_l} (q\rho_k^{-1} \rho_l, t\rho_k^{-1} \rho_l; q)_{\lambda_l - \lambda_k}}{(t^{-1}q\rho_k \rho_l, \rho_k \rho_l; q)_{\lambda_k + \lambda_l} (t^{-1}q\rho_k^{-1} \rho_l, \rho_k^{-1} \rho_l; q)_{\lambda_l - \lambda_k}}$$

where  $\rho_i := t_0 t^{i-1}$  and set for  $r \in \mathbb{Z}_+$ ,

$$(2.5.2) \quad K_r(\underline{t}; t) := \prod_{i=1}^r \frac{(\rho_i^{-2}; q)_\infty}{(q, \rho_i t_1, \rho_i^{-1} t_1, \rho_i t_2, \rho_i^{-1} t_2, \rho_i t_3, \rho_i^{-1} t_3; q)_\infty} \\ \cdot \prod_{1 \leq k < l \leq r} (\rho_k^{-1} \rho_l, \rho_k^{-1} \rho_l^{-1}; q)_\tau \\ = \prod_{i=1}^r \frac{(\rho_i^{-2}, t, t_0^{-2} t^{2-i-r}; q)_\infty}{(q, \rho_i t_1, \rho_i^{-1} t_1, \rho_i t_2, \rho_i^{-1} t_2, \rho_i t_3, \rho_i^{-1} t_3, t^i, t_0^{-2} t^{2-2i}; q)_\infty}$$

where  $t = q^\tau$ . The discrete weights  $\Delta^{(d)}(\rho q^\lambda; \underline{t}; t)$  (2.4.7) can now be rewritten as follows.

PROPOSITION 2.5.1. *For  $\lambda \in P(r)$  we have*

$$\Delta^{(d)}(\rho q^\lambda; \underline{t}; t) = K_r(\underline{t}; t) \Delta^{qR}(\rho q^\lambda; \underline{t}; t),$$

where  $\rho_i = t_0 t^{i-1}$ .

PROOF. We rewrite every factor  $(aq^m; q)_b$  in the explicit expression of the discrete weight  $\Delta^{(d)}(\rho q^\lambda; \underline{t}; t)$  in which  $m$  only depends on  $\lambda$  as a quotient of infinite products using (1.5.1). Then replace the factors of the form  $(cq^m; q)_\infty$  by  $(c; q)_\infty (c; q)_m^{-1}$  if  $m \in \mathbb{Z}_+$ , respectively by  $(c; q)_\infty (-c)^{-m} q^{-\binom{1-m}{2}} (qc^{-1}; q)_{-m}$  if  $m \in -\mathbb{N}$ . For the case  $m \in -\mathbb{N}$ , we used here the formula

$$(2.5.3) \quad (q^{1-l} x; q)_l = (-x)^l q^{-\binom{l}{2}} (x^{-1}; q)_l, \quad (l \in \mathbb{N}).$$

Using this method we obtain for  $j \in [1, r]$ ,

$$\begin{aligned}
(2.5.4) \quad & \prod_{i=1}^{j-1} \frac{1}{(\rho_i \rho_j q^{\lambda_{i-1} + \lambda_j}, \rho_i \rho_j^{-1} q^{\lambda_{i-1} - \lambda_j}; q)_{\lambda_i - \lambda_{i-1}}} \\
& = (-1)^{\lambda_{j-1}} t^{(j-1)\lambda_j} q^{\lambda_{j-1} - (\lambda_j - \lambda_{j-1}) + \binom{\lambda_j}{2}} \frac{(q; q)_{\lambda_j - \lambda_{j-1}} (t_0 \rho_j; q)_{\lambda_j}}{(\rho_j^2; q)_{\lambda_j + \lambda_{j-1}} (qt_0^{-1} \rho_j; q)_{\lambda_j}} \\
& \quad \cdot \prod_{i=1}^{j-1} \frac{(t \rho_i \rho_j; q)_{\lambda_i + \lambda_j} (q \rho_i^{-1} \rho_j; q)_{\lambda_j - \lambda_i}}{(\rho_i \rho_j; q)_{\lambda_i + \lambda_j} (qt^{-1} \rho_i^{-1} \rho_j; q)_{\lambda_j - \lambda_i}} t^{\lambda_i - \lambda_j}.
\end{aligned}$$

Using (2.5.4) and applying the same method to the explicit expression for the weight  $w_d$  (2.2.7) gives

$$\begin{aligned}
(2.5.5) \quad & w_d(\rho_j q^{\lambda_j}; \rho_j q^{\lambda_{j-1}}) \prod_{i=1}^{j-1} \frac{1}{(\rho_i \rho_j q^{\lambda_{i-1} + \lambda_j}, \rho_i \rho_j^{-1} q^{\lambda_{i-1} - \lambda_j}; q)_{\lambda_i - \lambda_{i-1}}} \\
& = \frac{(\rho_j^{-2}; q)_{\infty}}{(q, \rho_j t_1, \rho_j^{-1} t_1, \rho_j t_2, \rho_j^{-1} t_2, \rho_j t_3, \rho_j^{-1} t_3; q)_{\infty}} \\
& \quad \cdot \frac{(q \rho_j^2; q)_{2\lambda_j}}{(\rho_j^2; q)_{2\lambda_j} (q^{-1} t_0 t_1 t_2 t_3)^{\lambda_j}} \prod_{k=0}^3 \frac{(t_k \rho_j; q)_{\lambda_j}}{(qt_k^{-1} \rho_j; q)_{\lambda_j}} \\
& \quad \cdot \prod_{i=1}^{j-1} \frac{(t \rho_i \rho_j; q)_{\lambda_i + \lambda_j} (q \rho_i^{-1} \rho_j; q)_{\lambda_j - \lambda_i}}{(\rho_i \rho_j; q)_{\lambda_i + \lambda_j} (qt^{-1} \rho_i^{-1} \rho_j; q)_{\lambda_j - \lambda_i}} t^{\lambda_i - \lambda_j}
\end{aligned}$$

for  $j \in [1, r]$ . Now again applying the above mentioned method, gives

$$\begin{aligned}
(2.5.6) \quad & \prod_{i=1}^{j-1} (\rho_i^{-1} \rho_j q^{\lambda_j - \lambda_i}, \rho_i^{-1} \rho_j^{-1} q^{-\lambda_i - \lambda_j}; q)_{\tau} \\
& = \prod_{i=1}^{j-1} (\rho_i^{-1} \rho_j, \rho_i^{-1} \rho_j^{-1}; q)_{\tau} \frac{(t \rho_i^{-1} \rho_j; q)_{\lambda_j - \lambda_i} (q \rho_i \rho_j; q)_{\lambda_i + \lambda_j}}{(\rho_i^{-1} \rho_j; q)_{\lambda_j - \lambda_i} (qt^{-1} \rho_i \rho_j; q)_{\lambda_i + \lambda_j}} t^{-\lambda_i - \lambda_j}
\end{aligned}$$

for  $j \in [1, r]$ , where  $t = q^{\tau}$ . Now the proposition follows by multiplying (2.5.5) and (2.5.6) and taking the product over  $j \in [1, r]$ .  $\square$

In the remainder of the section we fix a  $N \in \mathbb{N}$ . In the next theorem the orthogonality relations for the Koornwinder polynomials are given when the parameters  $(\underline{t}, t)$  satisfy the truncation condition  $t^{n-1} t_0 t_3 = q^{-N}$ . The theorem will be formulated with the parameters considered as indeterminates. Set  $F := \mathbb{C}(\underline{t}, t)$ ,  $\mathbb{F} := \mathbb{C}(t_0, t_1, t_2, t)$  and  $\underline{t}_N := (t_0, t_1, t_2, t^{1-n} t_0^{-1} q^{-N})$ . Let  $A_F^{\mathcal{W}}$  respectively  $A_{\mathbb{F}}^{\mathcal{W}}$  be the algebra of  $\mathcal{W}$ -invariant Laurent polynomials over the field  $F$  respectively  $\mathbb{F}$ . Define the *Koornwinder polynomial*

$P_\lambda(\cdot; \underline{t}; t) \in A_F^{\mathcal{W}}$  of degree  $\lambda$  over the field  $F$  by

$$(2.5.7) \quad P_\lambda(\cdot; \underline{t}; t) := \left( \prod_{\mu < \lambda} \frac{D_{\underline{t}, t} - E_\mu(\underline{t}, t)}{E_\lambda(\underline{t}, t) - E_\mu(\underline{t}, t)} \right) m_\lambda.$$

Observe that  $P_\lambda(\cdot; \underline{t}_N; t) \in A_{\mathbb{F}}^{\mathcal{W}}$  is well defined since the eigenvalues  $\{E(\lambda; \underline{t}_N; t)\}_{\lambda \in \Lambda}$  (2.3.16) are mutually different as elements in  $\mathbb{C}[t_0, t_1, t_2, t]$ . The Koornwinder polynomial  $P_\lambda(\cdot; \underline{t}; t) \in A^{\mathcal{W}}$  of degree  $\lambda$  as defined in Theorem 2.3.7 (cf. Definition 2.3.8) can be reobtained from (2.5.7) by specializing the parameters  $(\underline{t}, t)$  to values in  $V \times (0, 1)$ .

**DEFINITION 2.5.2.**  $\{P_\lambda(\cdot; \underline{t}_N; t)\}_{\lambda \in \Lambda_N} \subset A_{\mathbb{F}}^{\mathcal{W}}$  with  $\Lambda_N := \{\lambda \in \Lambda \mid \lambda_1 \leq N\}$  are called the *multivariable (BC type)  $q$ -Racah polynomials*.

Let  $\lambda \in P(n)$ , then the weight  $\Delta^{qR}(\rho q^\lambda; \underline{t}_N; t) \in \mathbb{F}$  is well defined ( $\Delta^{qR}$  given by (2.5.1)) and it is non-zero if and only if  $\lambda_n \leq N$  due to the factor  $(\rho_n t_3; q)_{\lambda_n}$  in the numerator of  $\Delta^{qR}(\rho q^\lambda; \underline{t}; t)$ . So the bilinear form

$$(2.5.8) \quad \langle f, g \rangle_{qR, \underline{t}_N, t} := \sum_{\lambda \in P(n)} f(\rho q^\lambda) g(\rho q^\lambda) \Delta^{qR}(\rho q^\lambda; \underline{t}_N; t), \quad f, g \in A_{\mathbb{F}}^{\mathcal{W}},$$

takes its values in the field  $\mathbb{F}$ . Let  $\mathcal{N}^{qR}(\lambda; \underline{t}; t)$  for  $\lambda \in \Lambda$  be given by

$$(2.5.9) \quad \mathcal{N}^{qR}(\lambda; \underline{t}; t) := \frac{\mathcal{N}(\lambda; \underline{t}; t)}{K_n(\underline{t}; t) 2^{n!}}$$

where  $\mathcal{N}(\lambda)$  (2.3.18) is the expression for the quadratic norms of the Koornwinder polynomial  $P_\lambda$ . Substitution of the explicit expressions for  $\mathcal{N}(\lambda)$  and  $K_n$  in (2.5.9) yields that  $\mathcal{N}^{qR}(\lambda; \underline{t}_N; t) \in \mathbb{F}$  and that  $\mathcal{N}^{qR}(\lambda; \underline{t}_N; t)$  is non-zero if and only if  $\lambda \in \Lambda_N$ .

**THEOREM 2.5.3.** *Let  $N \in \mathbb{N}$ . The  $q$ -Racah polynomials  $P_\lambda(\cdot; \underline{t}_N; t)$  ( $\lambda \in \Lambda_N$ ) are orthogonal with respect to  $\langle \cdot, \cdot \rangle_{qR, \underline{t}_N, t}$  and the quadratic norms are given by*

$$(2.5.10) \quad \langle P_\lambda(\cdot; \underline{t}_N; t), P_\lambda(\cdot; \underline{t}_N; t) \rangle_{qR, \underline{t}_N, t} = \mathcal{N}^{qR}(\lambda; \underline{t}_N; t) \quad (\lambda \in \Lambda_N).$$

**PROOF.** Let  $\tilde{V} \subset (\mathbb{C}^*)^4$  be the set of parameters  $\underline{t} \in (\mathbb{C}^*)^4$  for which  $t_0 t_1 t_2 t_3 \in \mathbb{C} \setminus \mathbb{R}$ . Note that there exists an open dense subset  $I_N \subset (0, 1)$  such that  $E_\lambda(\underline{t}; t) \neq E_\mu(\underline{t}; t)$  for all  $\underline{t} \in \tilde{V}$ ,  $t \in I_N$  and all  $\lambda, \mu \in \Lambda_N$  with  $\lambda \neq \mu$ .

Fix  $t_0, t_1, t_2 \in \mathbb{C}^*$  such that  $\#\{\arg(t_i), \arg(t_i^{-1}) \mid i = 0, 1, 2\} = 6$  and  $t \in I_N$ . Then  $\underline{t}_N \in \tilde{V}$  and there exists a sequence  $\{t_{3,i}\}_{i \in \mathbb{Z}_+} \subset \mathbb{C}^*$  converging to  $t^{1-n} t_0^{-1} q^{-N}$  such that  $\underline{t}_i := (t_0, t_1, t_2, t_{3,i}) \in V \cap \tilde{V}$  for all  $i$  ( $V$  given in Definition 2.3.1). By considering a subsequence if necessary, we may assume that there exist  $(n, t_0)$ -residue pairs  $(C_i, \mathcal{C})$  where  $C_i$  is a  $\underline{t}_i$ -contour and where  $\mathcal{C}$  is a deformed circle such that the sequences  $\{t_1 q^j, t_2 q^j, t_{3,i} q^j\}_{j \in \mathbb{Z}_+}$  are in the interior of  $\mathcal{C}$  for all  $i$  and such that  $t^{n-1} t_0 q^N$  is in the exterior of  $\mathcal{C}$ . Then it follows from Theorem 2.3.7, Proposition 2.4.3 and Proposition

2.5.1 that

$$(2.5.11) \quad \frac{\mathcal{N}(\lambda; \underline{t}_i; t)}{K_n(\underline{t}_i; t)} \delta_{\lambda, \mu} = \frac{1}{(2\pi i)^n} \iint_{z \in \mathbb{C}^n} (P_\lambda P_\mu)(z; \underline{t}_i; t) \frac{\Delta(z; \underline{t}_i; t)}{K_n(\underline{t}_i; t)} \frac{dz}{z} \\ + \sum_{r=1}^n \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \sum_{\omega \in D(r)} \iint_{z \in \mathbb{C}^{n-r}} (P_\lambda P_\mu)(\omega, z; \underline{t}_i; t) \frac{\Delta_r(\omega, z; \underline{t}_i; t)}{K_n(\underline{t}_i; t)} \frac{dz}{z}$$

where  $\delta_{\lambda, \mu}$  is the Kronecker-delta and  $D(r) = D(r; C_i, \mathbb{C}; t_0; t)$  (2.4.6) (which is independent of  $i$ ). By (2.4.10) and Proposition 2.5.1 we have

$$(2.5.12) \quad \Delta_r(\omega, z; \underline{t}_i; t) = K_r(\underline{t}_i; t) \Delta^{qR}(\omega; \underline{t}_i; t) \Delta(z; \underline{t}_i; t) \delta_c(\omega; z).$$

After substitution of (2.5.12) in the right hand side of (2.5.11) for all  $r$ , it follows from the bounded convergence Theorem that the limit  $i \rightarrow \infty$  may be pulled through the integrals in the right hand side of (2.5.11). Only the completely discrete part survives the limit  $i \rightarrow \infty$  in the equality (2.5.11) since

$$\lim_{i \rightarrow \infty} \frac{K_r(\underline{t}_i; t)}{K_n(\underline{t}_i; t)} = 0, \quad 0 \leq r < n$$

by the factor  $(\rho_n t_3; q)_\infty$  in the denominator of  $K_n(\underline{t}_i; t)$ . The theorem follows now for the specialized parameter values  $t_0, t_1, t_3, t$  from the fact that

$$\{\rho q^\lambda \mid \lambda \in P(n), \lambda_n \leq N\} \subset D(n)$$

and the fact that  $\Delta^{qR}(\rho q^\lambda; \underline{t}_N; t) = 0$  for  $\lambda \in P(n)$  with  $\lambda_n > N$ . It is now clear that the theorem also holds over the field  $\mathbb{F}$ .  $\square$

The constant term identity can be simplified as follows.

COROLLARY 2.5.4. *For  $N \in \mathbb{N}$  we have the summation formula*

$$(2.5.13) \quad \langle 1, 1 \rangle_{qR, \underline{t}_N, t} = \prod_{i=1}^n \frac{(qt_0^2 t^{2n-i-1}, qt_1^{-1} t_2^{-1} t^{i-n}; q)_N}{(qt_0 t_1^{-1} t^{n-i}, qt_0 t_2^{-1} t^{n-i}; q)_N}.$$

PROOF. First note that by (2.3.25), (2.5.2) and (2.5.9) we have the explicit formula

$$(2.5.14) \quad \mathcal{N}^{qR}(0; \underline{t}; t) = \prod_{i=1}^n \frac{(t_0 t_1 t_2 t_3 t^{2n-i-1}, t_0^{-1} t_1 t^{1-i}, t_0^{-1} t_2 t^{1-i}, t_0^{-1} t_3 t^{1-i}; q)_\infty}{(t_0^{-2} t^{1+i-2n}, t_1 t_2 t^{i-1}, t_1 t_3 t^{i-1}, t_2 t_3 t^{i-1}; q)_\infty}.$$

Then (2.5.13) follows by substitution of  $t_3 = t_0^{-1} t^{1-n} q^{-N}$  in (2.5.14) and by applying formula (2.5.3) repeatedly (see also [18, Section 2.3]).  $\square$

The second order  $q$ -difference operator  $D_{\underline{t}_N, t}$  (2.3.13) diagonalizes the  $q$ -Racah polynomials  $\{P_\lambda(\cdot; \underline{t}_N; t)\}_{\lambda \in \Lambda_N}$ . By Theorem 2.5.3 we conclude that  $D_{\underline{t}_N, t}$  is symmetric with respect to  $\langle \cdot, \cdot \rangle_{qR, \underline{t}_N, t}$ . In [20] the symmetry of  $D_{\underline{t}_N, t}$  was proved by direct calculations and the orthogonality relations for the multivariable  $q$ -Racah polynomials were proved using the symmetry of  $D_{\underline{t}_N, t}$ . Furthermore, in [20] the quadratic norms of the

$q$ -Racah polynomials were expressed in terms of the quadratic norm of the unit polynomial by studying Pieri formulas for the  $q$ -Racah polynomials. The constant term identity (2.5.13) was recently proved by van Diejen [18, Theorem 3] by truncating a multivariable analogue of Roger's  ${}_6\phi_5$ -series [18, Theorem 2], which in turn is closely related to an Aomoto-Ito type sum (cf. [4], [44]) for the non-reduced root system  $BC_n$ . The proofs of the summation formulas in [18] are based on a multiple  ${}_6\psi_6$  summation formula of Gustafson.

In the one variable case it is known that the Askey-Wilson integral can be rewritten as an infinite sum of residues for some parameter region by shifting the contour over four infinite sequences of poles (see [7, Theorem 2.1]). More generally one can ask the question whether a completely discrete orthogonality measure for the Koornwinder polynomials can be obtained by pulling the  $t$ -contours over certain infinite sequence of poles in the orthogonality relations of the Koornwinder polynomials (Theorem 2.3.7).

Strong indications in that direction can be found in Gustafson's paper [32] and the recent paper of Tarasov and Varchenko [127] where contours in multidimensional integrals are shifted over infinite sequences of poles in order to arrive at (purely discrete) multidimensional Jackson integrals. Another strong indication is the fact that the Macdonald polynomials are orthogonal with respect to Aomoto-Ito type (cf. [4], [44]) weight functions (see Cherednik [15]). Since the  $B$ ,  $C$  and  $D$  type Macdonald polynomials can be obtained from the Koornwinder polynomials by suitable specialization of the parameters we thus have orthogonality relations for these subfamilies of the Koornwinder polynomials with respect to infinite discrete measures (and the corresponding discrete weights are directly related to (2.5.1), see [18]).

We will not consider here the above mentioned questions. In the next chapter we will look instead at the implications of the residue calculus for certain limit cases of the Koornwinder polynomials (multivariable big and little  $q$ -Jacobi polynomials). In order to study these limit cases we first need to consider the Koornwinder polynomials for yet another parameter domain. This will be the subject of the next section.

## 2.6. Koornwinder polynomials with positive orthogonality measure

In this section the Koornwinder polynomials are considered for parameters  $\underline{t}$  in the following parameter domain.

DEFINITION 2.6.1. *Let  $V_K$  be the set of parameters  $\underline{t} = (t_0, t_1, t_2, t_3)$  which satisfy the following conditions:*

- (1) *The parameters  $t_0, t_1, t_2, t_3$  are real, or if complex, then they appear in conjugate pairs.*
- (2)  *$t_k t_l \notin \mathbb{R}_{\geq 1}$  for all  $0 \leq k < l \leq 3$ .*

Observe that parameters  $\underline{t} \in V_K$  satisfy the following properties:

- (A)  $t_i \in \mathbb{R}$  if  $|t_i| \geq 1$ ;
- (B) There are at most two parameters with modulus  $\geq 1$ . If there are two, then one is positive and the other is negative.



It will be shown that the multivariable Koornwinder polynomials are orthogonal with respect to a positive, partly discrete orthogonality measure for  $t \in (0, 1)$  and  $\underline{t} \in V_K$  by shifting the contour  $C^n$  in the integral

$$(2.6.1) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^n} P_\lambda(z) P_\mu(z) \Delta(z) \frac{dz}{z}$$

to the  $n$ -torus  $T^n$  for a specific parameter domain  $V_0 \subset V$  (here  $C$  is a  $\underline{t}$ -contour (Definition 2.3.2) and  $V$  is the parameter domain given in Definition 2.3.1). Then a partly discrete orthogonality measure will be obtained which turns out to be well defined and positive for parameter values  $\underline{t} \in V_K$ . Orthogonality relations for parameter values  $\underline{t} \in V_K$  with respect to this positive, partly discrete orthogonality measure can then be derived by suitable continuity arguments.

The parameter domain  $V_0$  is defined as follows.

DEFINITION 2.6.2. *Let  $V_0$  be the set of parameters  $\underline{t} \in V$  for which*

- (i) *at most two parameters have modulus  $> 1$ ;*
- (ii)  *$t_i t^j q^p \notin T$  for  $i \in [0, 3]$ ,  $j \in [-1, n-1]$  and  $p \in \mathbb{Z}$ .*

Fix  $t \in (0, 1)$ ,  $\underline{t} \in V_0$  and  $0 \leq i \neq j \leq 3$  such that  $|t_k| < 1$  for  $k \neq i, j$ . Write  $\rho_p^{(i)} := t^{p-1} t_i$  respectively  $\rho_p^{(j)} := t^{p-1} t_j$  for  $p \in \mathbb{Z}$ . Define for  $r \in \mathbb{N}$  a finite discrete set  $D_i(r) = D_i(r; \underline{t}; t) \subset \mathbb{C}^r$  by

$$(2.6.2) \quad D_i(r) := \{\rho^{(i)} q^\mu \mid \mu \in P(r), |\rho_r^{(i)} q^{\mu_r}| > 1\}$$

and similarly for  $D_j(r)$ , where  $P(r)$  is given by (2.4.4) and

$$\rho^{(i)} q^\mu = (\rho_1^{(i)} q^{\mu_1}, \dots, \rho_r^{(i)} q^{\mu_r})$$

for  $\mu \in P(r)$ . Observe that  $D_i(r) = \emptyset$  if  $|t_i| < 1$ . Furthermore, write  $F(r) = F(r; \underline{t}; t) \subset \mathbb{C}^r$  for the disjoint union

$$(2.6.3) \quad F(r) := \bigcup_{\substack{l+m=r \\ l, m \in \mathbb{Z}_+}} D_i(l) \times D_j(m) \quad (r \in [1, n]).$$

Here the convention is used that  $D_i(l) \times D_j(m) = \emptyset$  if  $l > 0$  and  $D_i(l) = \emptyset$  or if  $m > 0$  and  $D_j(m) = \emptyset$ , and that  $D_i(0) \times D_j(m) = D_j(m)$ ,  $D_i(l) \times D_j(0) = D_i(l)$ . Let  $\omega \in F(r)$ ,  $z \in T^{n-r}$  and set

$$(2.6.4) \quad d\nu_r^K(\omega, z; \underline{t}; t) := \Delta_r^K(\omega, z; \underline{t}; t) \frac{dz}{z}$$

with weight function  $\Delta_r^K(\omega, z)$  for  $\omega = (\vartheta, \zeta) \in D_i(l) \times D_j(m)$  given by

$$(2.6.5) \quad \Delta_r^K(\vartheta, \zeta, z; \underline{t}; t) := \Delta^{(d)}(\vartheta; t_i) \Delta^{(d)}(\zeta; t_j) \Delta(z; \underline{t}; t) \delta_c(\vartheta; \zeta, z) \delta_c(\zeta; z)$$

where  $\Delta^{(d)}$  is given by (2.4.7) and  $\delta_c$  is given by (2.4.11). For the special case  $l = 0$  respectively  $m = 0$ , (2.6.5) simplifies to

$$(2.6.6) \quad \Delta_r^K(\zeta, z; \underline{t}; t) = \Delta^{(d)}(\zeta; t_j) \Delta(z; \underline{t}; t) \delta_c(\zeta; z) = \Delta_r(\zeta, z; t_j)$$

respectively

$$(2.6.7) \quad \Delta_r^K(\vartheta, z; \underline{t}; t) = \Delta^{(d)}(\vartheta; t_i) \Delta(z; \underline{t}; t) \delta_c(\vartheta; z) = \Delta_r(\vartheta, z; t_i).$$

where  $\Delta_r$  is given by (2.4.10). The following lemma is an easy consequence of the residue calculus developed in Section 2.4.

LEMMA 2.6.3. *Let  $t \in (0, 1)$  and  $\underline{t} \in V_0$ . Let  $C$  be a  $\underline{t}$ -contour and  $f \in A^{\mathcal{W}}$ . Then,*

$$(2.6.8) \quad \frac{1}{(2\pi i)^n} \iint_{z \in C^n} f(z) d\nu(z) = \frac{1}{(2\pi i)^n} \iint_{z \in T^n} f(z) d\nu(z) + \sum_{r=1}^n \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \sum_{\omega \in F(r)} \iint_{z \in T^{n-r}} f(\omega, z) d\nu_r^K(\omega, z).$$

PROOF. If  $|t_k| < 1$  for all  $k$  then we only have the completely continuous measure  $d\nu$  on  $T^n$  in the right hand side of (2.6.8) since  $F(r) = \emptyset$ . Since  $T$  is a  $\underline{t}$ -contour in this case, the lemma follows from Lemma 2.3.4.

Suppose that at most one parameter has modulus  $> 1$ . By the symmetry of  $d\nu(z; \underline{t}; t)$  in the four parameters  $\underline{t}$ , we may assume that  $|t_0| > 1$ . By Lemma 2.3.4, we may assume that the  $\underline{t}$ -contour  $C$  satisfies the additional conditions that  $A^+ := \{x \in [0, 1] \mid r_C(x) > 1\}$  is an open interval and that  $\alpha_0^+ \in A^+$  but  $\alpha_i^\pm \notin A^+$  for  $i = 1, 2, 3$  (here  $r_C$  is as in Definition 2.3.2, and  $\alpha_i^+$  is given by (2.3.2)). Then  $(C, T)$  is a  $(n, t_0)$ -residue pair since  $\underline{t} \in V_0$  (Definition 2.6.2), and  $D_0(l) = D(l; C, T; t_0)$  (2.4.6) since  $t_0$  is in the interior of  $C$ . The lemma is then a direct consequence of Proposition 2.4.3 and (2.6.7).

Suppose now that two parameters have moduli  $> 1$ . Without loss of generality, we may assume that  $|t_0| > 1$  and  $|t_1| > 1$ , and that the  $\underline{t}$ -contour  $C$  satisfies the additional condition that

$$\{x \in [0, 1] \mid r_C(x) > 1\} = A_0^+ \cup A_1^+$$

disjoint union, with the  $A_i^+$  open intervals such that  $\alpha_i^+ \in A_i^+$  and  $\alpha_j^\pm \notin A_i^+$  for  $j \neq i$  and  $i = 0, 1$ . Let  $C' := \phi_{C'}([0, 1])$  be the deformed circle with parametrization  $\phi_{C'}(x) = r_{C'}(x)e^{2\pi i x}$  given by

$$r_{C'}(x) := r_C(x) \quad (x \notin A_0^+ \cup A_0^-), \quad r_{C'}(x) := 1 \quad (x \in A_0^+ \cup A_0^-),$$

where  $A_0^- := (1 - \beta, 1 - \alpha)$  when  $A_0^+ = (\alpha, \beta)$ . Then  $(C, C')$  is a  $(n, t_0)$ -residue pair,  $(C', T)$  is a  $(n, t_1)$ -residue pair and  $D_0(l) = D(l; C, C'; t_0)$  respectively  $D_1(m) = D(m; C', T; t_1)$  since  $t_0$  and  $t_1$  are in the interior of  $C$ . Write  $\Omega'$  for the domain associated with  $(C', T)$ , then  $\delta_c(\vartheta; \cdot) \in \mathcal{O}_{\mathcal{W}}((\Omega')^{n-l})$  for  $\vartheta \in D_0(l)$ , hence the lemma follows by applying Proposition 2.4.3 first to the  $(n, t_0)$ -residue pair  $(C, C')$ , and then to the  $(n, t_1)$ -residue pair  $(C', T)$ .  $\square$

For  $t = q^k$  with  $k \in \mathbb{N}$ , formula (2.6.8) can be rewritten as

$$(2.6.9) \quad \frac{1}{(2\pi i)^n} \iint_{z \in \mathbb{C}^n} f(z) d\nu(z) = \sum_{r=0}^n \frac{2^r \binom{n}{r}}{(2\pi i)^{n-r}} \sum_{e_1, \dots, e_r} \sum_{\substack{z_i \in \{e_i q^l\}_{l=0}^{N_{e_i}} \\ i=1, \dots, r}} \\ \iint_{(z_{r+1}, \dots, z_n) \in T^{n-r}} f(z) \delta(z; q^k) \prod_{i=1}^r w_d(z_i; e_i) \prod_{j=r+1}^n w_c(z_j) \frac{dz_j}{z_j}$$

for  $f \in A^{\mathcal{W}}$ , where the second sum is over  $e_i \in \{t_j \mid j \in [0, 3], |t_j| > 1\}$  and  $N_{e_i}$  is the largest positive integer such that  $|e_i q^{N_{e_i}}| > 1$ . Formula (2.6.9) can also be proved directly by induction on  $n$  using similar arguments as in the proof of Lemma 2.4.2.

Define now bilinear forms  $\langle \cdot, \cdot \rangle_{r, \underline{t}, t}$  on  $A^{\mathcal{W}}$  for  $r \in [0, n]$ ,  $\underline{t} \in V_0$  and  $t \in (0, 1)$  by

$$(2.6.10) \quad \begin{aligned} \langle f, g \rangle_0 &:= \iint_{z \in T^n} f(z) g(z) d\nu(z), \\ \langle f, g \rangle_r &:= \sum_{\omega \in F(r)} \iint_{z \in T^{n-r}} f(\omega, z) g(\omega, z) d\nu_r^K(\omega, z), \quad r \in [1, n] \end{aligned}$$

for  $f, g \in A^{\mathcal{W}}$  and set

$$(2.6.11) \quad \langle f, g \rangle_{\underline{t}, t} := \sum_{r=0}^n \frac{2^r (n-r+1)}{(2\pi i)^{n-r}} \langle f, g \rangle_{r, \underline{t}, t}, \quad f, g \in A^{\mathcal{W}}.$$

In the following lemma the symmetric bilinear form  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  is considered for parameter values  $(\underline{t}, t) \in V_K \times (0, 1)$ .

LEMMA 2.6.4. *Let  $t \in (0, 1)$  and  $\underline{t} \in V_K$ .*

- (i) *The bilinear form  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  is well defined;*
- (ii) *The weight function  $\Delta(z; \underline{t}; t)$  respectively  $\Delta_r^K(\omega, z; \underline{t}; t)$  is positive for  $z \in T^n$  respectively  $(\omega, z) \in F(r) \times T^{n-r}$  ( $r \in [1, n]$ ).*

PROOF. The discrete weights  $w_d$  (2.2.7) appearing as factors of the weight function  $\Delta_r^K(\omega, z)$  for  $r > 0$  are well defined and strictly positive. Indeed if  $t_0 t_1 t_2 t_3 = 0$ , then the factors  $(t_i q/t_j; q)_k t_j^k$  in the denominator of  $w_d$  should be read as  $\prod_{l=0}^{k-1} (t_j - t_i q^{l+1})$ . The factor  $\delta(z; t) = |\delta_+(z; t)|^2$  is also well defined and positive for  $z \in T^{n-r}$ .

Without loss of generality we may assume that  $|t_2|, |t_3| < 1$ . Fix  $\omega = (\vartheta, \zeta) \in F(r)$  with  $\vartheta \in D_0(l)$  and  $\zeta \in D_1(m)$  ( $r = l + m$ ). The factor  $\delta_d(\vartheta)$  respectively  $\delta_d(\zeta)$  (2.4.8) appearing in the discrete weights  $\Delta^{(d)}(\vartheta; t_0)$  respectively  $\Delta^{(d)}(\zeta; t_1)$  when  $l > 0$  respectively  $m > 0$  is well defined and strict positive. Indeed, if  $\vartheta \in D_0(l)$  and  $l > 0$ , then  $|t_0| > 1$ , hence  $t_0 \in \mathbb{R}$ . Then  $\delta_d(\vartheta) > 0$  follows easily from the definition of the set  $D_0(l)$  (2.6.2).

It remains to show that  $h(z) := (\prod_{l=1}^{n-r} w_c(z_l; \underline{t})) \delta_c(\vartheta; \zeta, z) \delta_c(\zeta; z)$  is well defined and positive for  $z \in T^{n-r}$ . We check the case that both  $t_0$  and  $t_1$  have moduli  $\geq 1$ , and

that  $t_0$  is positive real and  $t_1$  negative real (see property **(B)** for parameters  $\underline{t} \in V_K$ ). The case that at most one parameter has modulus  $\geq 1$  will then also be clear.

Rewrite the factor  $(x^2, x^{-2}; q)_\infty$  appearing in the numerator of  $w_c(x; \underline{t})$  as

$$(x^2, x^{-2}; q)_\infty = (x, -x, x^{-1}, -x^{-1}; q)_\infty (qx^2, qx^{-2}; q^2)_\infty$$

then it is sufficient to check that the factors of the form

$$h_0(x) := \frac{(x, x^{-1}; q)_\infty}{(t_0 x, t_0 x^{-1}; q)_\infty} \prod_{k=1}^l (\vartheta_k x, \vartheta_k x^{-1}, \vartheta_k^{-1} x, \vartheta_k^{-1} x^{-1}; q)_\tau,$$

$$h_1(x) := \frac{(-x, -x^{-1}; q)_\infty}{(t_1 x, t_1 x^{-1}; q)_\infty} \prod_{k=1}^m (\zeta_k x, \zeta_k x^{-1}, \zeta_k^{-1} x, \zeta_k^{-1} x^{-1}; q)_\tau$$

( $l, m \in [0, n-1]$ ) are well defined and positive for  $x \in T$  (here  $t = q^\tau$ ). Indeed, the remaining factors of  $w_c(x; \underline{t})$  are easily seen to be well defined and positive since  $|t_2|, |t_3| < 1$  and  $t_2, t_3$  are both real or are a conjugate pair, while the remaining factors

$$\prod_{\varepsilon_i, \varepsilon_j = \pm 1} (\vartheta_i^{\varepsilon_i} \zeta_j^{\varepsilon_j}; q)_\tau \quad (i \in [1, l], j \in [1, m])$$

of  $\delta_c(\vartheta; \zeta, z)$  are well defined and positive since  $t_0$  is positive real and  $t_1$  is negative real. Now let  $\lambda \in P(l)$  such that  $\vartheta = \rho^{(0)} q^\lambda \in D_0(l)$ , then  $h_0(x) = |h_0^+(x)|^2$  for  $x \in T$  with  $h_0^+$  given by

$$h_0^+(x) := \frac{(x; q)_\infty}{(t_0 x; q)_\infty} \prod_{k=1}^l (\vartheta_k x, \vartheta_k^{-1} x; q)_\tau$$

$$= \frac{(x; q)_\infty}{(t \vartheta_l x; q)_\infty} \prod_{k=1}^l \frac{(\vartheta_k^{-1} x; q)_\tau}{(t \vartheta_{k-1} x; q)_{\lambda_k - \lambda_{k-1}}}$$

where  $\vartheta_0 := t^{-1} t_0$  and  $\lambda_0 := 0$ . It follows that  $h_0^+(x)$  is well defined for  $x \in T$ , since the possible zero at  $x = 1$  of the factor  $(t \vartheta_l x; q)_\infty$  in the denominator can be compensated by the zero at  $x = 1$  of the factor  $(x; q)_\infty$ . Similarly, one deals with  $h_1(x)$ .  $\square$

We write  $A_{\mathbb{R}}^{\mathcal{W}}$  for the  $\mathbb{R}$ -algebra of  $\mathcal{W}$ -invariant Laurent polynomials in the variables  $z_1, \dots, z_n$ . The following corollary is a direct consequence of Lemma 2.6.4.

**COROLLARY 2.6.5.** *Let  $\underline{t} \in V_K$  and  $t \in (0, 1)$ . Then the restriction of the bilinear form  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  to  $A_{\mathbb{R}}^{\mathcal{W}} \times A_{\mathbb{R}}^{\mathcal{W}}$  maps into  $\mathbb{R}$  and is positive definite.*

**PROOF.** The monomials  $m_\lambda$  ( $\lambda \in \Lambda$ ) are real-valued on  $F(r) \times T^{n-r}$  since  $F(r) \subset \mathbb{R}^r$  by property **(A)** for parameters in  $V_K$  (Definition 2.6.1), so the assertion follows from Lemma 2.6.4(ii).  $\square$

The following theorem defines the Koornwinder polynomials for parameters  $\underline{t} \in V_K$  and  $t \in (0, 1)$  as a special choice of orthogonal basis for  $A_{\mathbb{R}}^{\mathcal{W}}$  with respect to the positive definite bilinear form  $\langle \cdot, \cdot \rangle_{\underline{t}, t} : A_{\mathbb{R}}^{\mathcal{W}} \times A_{\mathbb{R}}^{\mathcal{W}} \rightarrow \mathbb{R}$ .

**THEOREM 2.6.6.** *For parameters  $(\underline{t}, t) \in V_K \times (0, 1)$  there exists a unique basis  $\{P_\lambda(\cdot; \underline{t}; t)\}_{\lambda \in \Lambda}$  of  $A_{\mathbb{R}}^{\mathcal{W}}$  such that*

$$(i) P_\lambda(\cdot; \underline{t}; t) = m_\lambda + \sum_{\mu < \lambda} c_{\lambda, \mu}(\underline{t}; t) m_\mu, \text{ some } c_{\lambda, \mu}(\underline{t}; t) \in \mathbb{R};$$

$$(ii) \langle P_\lambda(\cdot; \underline{t}; t), P_\mu(\cdot; \underline{t}; t) \rangle_{\underline{t}, t} = 0 \text{ if } \mu \neq \lambda.$$

*Furthermore,  $P_\lambda(\cdot; \underline{t}; t)$  is an eigenfunction of  $D_{\underline{t}, t}$  with eigenvalue  $E_\lambda(\underline{t}; t)$  and we have the explicit evaluation formula*

$$\langle P_\lambda(\cdot; \underline{t}; t), P_\lambda(\cdot; \underline{t}; t) \rangle_{\underline{t}, t} = \mathcal{N}(\lambda; \underline{t}; t), \quad \lambda \in \Lambda$$

*for the quadratic norms of the polynomials  $P_\lambda$ .*

**PROOF.** Fix  $t \in (0, 1)$  and  $\underline{t} \in V_K$ . Since  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  is positive definite on  $A_{\mathbb{R}}^{\mathcal{W}}$ , there exists for  $\lambda \in \Lambda$  a unique  $\mathcal{W}$ -invariant Laurent polynomial  $P_\lambda(\cdot; \underline{t}; t) \in A_{\mathbb{R}}^{\mathcal{W}}$  satisfying (i) and the conditions  $\langle P_\lambda(\cdot; \underline{t}; t), m_\mu \rangle_{\underline{t}, t} = 0$  for all  $\mu < \lambda$ . Furthermore,

$$(2.6.12) \quad P_\lambda(z; \underline{t}; t) = m_\lambda(z) - \sum_{\mu < \lambda} \frac{\langle m_\lambda, P_\mu(\cdot; \underline{t}; t) \rangle_{\underline{t}, t}}{\langle P_\mu(\cdot; \underline{t}; t), P_\mu(\cdot; \underline{t}; t) \rangle_{\underline{t}, t}} P_\mu(z; \underline{t}; t), \quad \lambda \in \Lambda.$$

The polynomials  $P_\lambda(z; \underline{t}; t) = m_\lambda(z) + \sum_{\mu < \lambda} c_{\lambda, \mu}(\underline{t}; t) m_\mu(z)$  as defined in Theorem 2.3.7 also satisfy the formula (2.6.12) when  $\underline{t} \in V_0$ , in view of Lemma 2.6.3. Fix  $\underline{t} \in V_K \setminus V_K^-$ , where  $V_K^-$  is the set of parameters  $\underline{t} \in V_K$  such that  $t_i = \pm t^{-m} q^{-s}$  for some  $i \in [0, 3]$ ,  $m \in [0, n-1]$  and  $s \in \mathbb{Z}_+$ . Let  $\{\underline{t}_k\}_{k \in \mathbb{Z}_+}$  be a sequence in  $V_0$  converging to  $\underline{t}$ . Then, by the bounded convergence Theorem,

$$(2.6.13) \quad \lim_{k \rightarrow \infty} \langle f, g \rangle_{\underline{t}_k, t} = \langle f, g \rangle_{\underline{t}, t}, \quad \forall f, g \in A^{\mathcal{W}}.$$

Indeed, by assuming  $\underline{t} \notin V_K^-$ , we have  $F(r; \underline{t}; t) = F(r; \underline{t}; t)$  for  $\underline{t}$  in an open neighbourhood of  $\underline{t}$  ( $r \in [1, n]$ ), and no zeros in the denominator of the expression for  $\Delta_r^K(\omega, \cdot; \underline{t}; t)$  ( $\omega \in F(r)$ ,  $r \in [0, n-1]$ ) occur which need to be compensated by zeros in the numerator (see the proof of Lemma 2.6.4). Hence the bounded convergence Theorem may be applied at once.

By induction on  $\lambda$  we then obtain from (2.6.12) and (2.6.13) that

$$(2.6.14) \quad \lim_{k \rightarrow \infty} c_{\lambda, \mu}(\underline{t}_k; t) = c_{\lambda, \mu}(\underline{t}; t), \quad \mu < \lambda,$$

where  $c_{\lambda, \mu}$  are the expansion coefficients of  $P_\lambda$  with respect to the monomials  $m_\mu$  ( $\mu \in \Lambda$ ). By the residue calculus given in Lemma 2.6.3, Theorem 2.3.7 can be reformulated with respect to the bilinear form  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  for  $\underline{t} \in V_0$ . The theorem follows then for  $\underline{t} \in V_K \setminus V_K^-$  by taking limits in the reformulated results using Proposition 2.3.6 and (2.6.14).

To prove the theorem for  $\underline{t} \in V_K^-$ , we use again a continuity argument. We treat here one typical example, the general case is derived similarly. Assume that  $\underline{t} \in V_K^-$  with  $t_0 = t^{-m} q^{-s}$  for some  $m \in [0, n-1]$ ,  $s \in \mathbb{Z}_+$  and that  $t_i \neq t^{-l} q^{-s'}$  for all  $i \in \{1, 2, 3\}$ ,  $l \in [0, n-1]$  and  $s' \in \mathbb{Z}_+$ . Then there exists an  $\varepsilon > 0$  such that  $(\tau_0, t_1, t_2, t_3) \in V_K \setminus V_K^-$  and  $F(r; \tau_0, t_1, t_2, t_3; t) = F(r; \underline{t}; t)$  for all  $r \in [1, n]$  and all  $\tau_0 \in \mathbb{R}_{>0}$  with  $t_0 - \tau_0 < \varepsilon$ . We claim that

$$(2.6.15) \quad \lim_{\tau_0 \uparrow t_0} \langle f, g \rangle_{\tau_0, t_1, t_2, t_3, t} = \langle f, g \rangle_{\underline{t}, t}, \quad \forall f, g \in A^{\mathcal{W}}.$$

We use the bounded convergence Theorem. In Lemma 2.6.4 it was shown that zeros in the denominator of the expression for the weight function  $\Delta_r^K(\omega, \cdot; \underline{t}; t)$  can occur when  $\omega \in F(r; \underline{t}; t)$  and  $\underline{t} \in V_K^-$ , and that these zeros can be compensated by zeros in the numerator. It follows from the specific form of these compensated zeros and from the fact that the functions

$$h^\pm(u, x) := \begin{cases} \frac{(1 \pm x)}{(1 \pm ux)} & \text{if } u \neq 1 \\ 1 & \text{if } u = 1 \end{cases}$$

are bounded on  $U \times T$  where  $U \subset \mathbb{R}_{>0}$  is some open set containing 1, that the bounded convergence Theorem may be applied in the limit (2.6.15). Now the theorem for the specific parameter values  $\underline{t}$  follows by continuity arguments from (2.6.15).  $\square$

For parameters  $\underline{t} \in V_K$  with  $|t_i| \leq 1$  the orthogonality measure is completely continuous (i.e.  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_0$ ) and coincides with Koornwinder's orthogonality measure [65]. In particular the orthogonality relations reduce to Koornwinder's orthogonality relations (see [65]) and the quadratic norm evaluations reduce to van Diejen's quadratic norm evaluations (see [17]) for parameter values  $\underline{t} \in V_K$  with  $|t_i| \leq 1$ .

Theorem 2.6.6 for  $n = 1$  reduces to the orthogonality relations and norm evaluations stated in [7, Theorem 2.5].

For  $t = q^k$  ( $k \in \mathbb{N}$ ) the bilinear form  $\langle \cdot, \cdot \rangle_{\underline{t}, q^k}$  (2.6.11) can be rewritten as

$$(2.6.16) \quad \langle f, g \rangle = \sum_{r=0}^n \frac{2^r \binom{n}{r}}{(2\pi i)^{n-r}} \sum_{e_1, \dots, e_r} \sum_{\substack{z_i \in \{e_i q^l\}_{l=0}^{N e_i} \\ i=1, \dots, r}} \iint_{(z_{r+1}, \dots, z_n) \in T^{n-r}} f(z)g(z)\delta(z; q^k) \prod_{i=1}^r w_d(z_i; e_i) \prod_{j=r+1}^n w_c(z_j) \frac{dz_j}{z_j}$$

for  $f, g \in A^{\mathcal{W}}$ . This is the form of the partly discrete orthogonality measure for the Koornwinder polynomials with discrete deformation parameter  $t = q^k$  which was studied earlier in [117] and [118].

## CHAPTER 3

# Limit transitions for multivariable orthogonal polynomials

### 3.1. Introduction

The one variable big and little  $q$ -Jacobi polynomials depend apart from  $q$  on (essentially) three and two parameters, respectively. The associated orthogonality measures are completely discrete and have infinitely many discrete mass points. The orthogonality measures can be expressed most conveniently in terms of Jackson integrals. The one variable big and little  $q$ -Jacobi polynomials are  $q$ -analogues of the classical Jacobi polynomials in the sense that when  $q$  tends to 1, the big and little  $q$ -Jacobi polynomials tend, up to a possible translation and dilation of the variable, to the classical Jacobi polynomials.

The families of one variable big and little  $q$ -Jacobi polynomials are members of the Askey tableau. The Askey tableau consists of families of (basic hypergeometric) orthogonal polynomials which are joint eigenfunctions of a second order  $q$ -difference operator. Some families can be obtained from others by limit transitions or by specializations of parameters. This induces the hierarchy structure between the families. From this point of view, the four parameter family of Askey-Wilson polynomials is on top of the hierarchy and the families of big and little  $q$ -Jacobi polynomials are directly below the Askey-Wilson polynomials. Suitable limit transitions are known from the Askey-Wilson polynomials to the big and little  $q$ -Jacobi polynomials (cf. [66]).

In this chapter the limit transitions from Askey-Wilson polynomials to big and little  $q$ -Jacobi polynomials are generalized to the multivariable setting. It is shown that in these limits, the positive, partly continuous orthogonality measure for the Koornwinder polynomials (cf. Section 2.6) tend to certain completely discrete measures. These discrete measures can be expressed most conveniently in terms of multidimensional Jackson integrals. They reduce to the orthogonality measures for the little and big  $q$ -Jacobi polynomials in the one variable setting. These discrete measures are used to define a four parameter family of multivariable big  $q$ -Jacobi polynomials and a three parameter family of multivariable little  $q$ -Jacobi polynomials. Orthogonality relations and quadratic norm evaluations for the multivariable big and little  $q$ -Jacobi polynomials then follow by taking the limits in the orthogonality relations and the quadratic norm evaluations for the Koornwinder polynomials (cf. previous chapter).

The constant term identities for the multivariable little and big  $q$ -Jacobi polynomials reduce to well-known  $q$ -analogues of the Selberg integral. The constant term identity

for the multivariable little  $q$ -Jacobi polynomials (Corollary 3.3.6) is known as the Askey-Habsieger-Kadell formula (see [5],[33],[54]) and was proved in full generality by Aomoto [4]. The constant term identity for the multivariable big  $q$ -Jacobi polynomials with one of the parameters discrete (Corollary 3.5.3) was conjectured by Askey [5] and proved by Evans [29]. The constant term identity in the general form (Corollary 3.4.8) is equivalent to Tarasov's and Varchenko's constant term identity [127, Theorem (E.10)].

In this chapter we develop also a more algebraic method for proving limit transitions between families of multivariable orthogonal polynomials. The method is based on the fact that the multivariable basic hypergeometric orthogonal polynomials under consideration can be characterized for generic parameter values as a special type of eigenfunction for an explicitly known second order  $q$ -difference operator. The proof of the limit transitions reduces then to the problem of computing limits of second order  $q$ -difference operators and their eigenvalues. Usually this method is less computational and valid for generic parameter values. The disadvantage of this method is that properties of the polynomials, such as orthogonality relations and quadratic norm evaluations, can not be rigorously transported to their limit cases. In this chapter we apply this algebraic method to prove limit transitions from Koornwinder polynomials, multivariable big and little  $q$ -Jacobi polynomials to generalized Jacobi polynomials.

This chapter is organized as follows. In Section 3.2 the one variable little and big  $q$ -Jacobi polynomials are defined as limit cases of the Askey-Wilson polynomials and their orthogonality relations and quadratic norm evaluations are recalled. In Section 3.3 respectively Section 3.4, the multivariable little respectively big  $q$ -Jacobi polynomials are introduced and studied. The proof of the limit from the orthogonality measure of the Koornwinder polynomials to the discrete orthogonality measures of the multivariable little respectively big  $q$ -Jacobi polynomials is postponed to Section 3.7 respectively Section 3.8. In Section 3.5 the orthogonality measures of the multivariable big and little  $q$ -Jacobi polynomials are considered when one parameter is discrete. Then it is shown that the constant term identities for the multivariable big and little  $q$ -Jacobi polynomials reduce to well-known identities which were already studied in the beginning of the 1980's, cf. [5]. In Section 3.6 the algebraic approach for proving limit transitions is developed.

### 3.2. One variable big and little $q$ -Jacobi polynomials

In this section the families of little and big  $q$ -Jacobi polynomials are introduced as limit cases of the Askey-Wilson polynomials. For the notations on Askey-Wilson polynomials, we refer the reader to Section 2.2.

The monic little  $q$ -Jacobi polynomials  $\{P_n^L(\cdot; a, b)\}_{n \in \mathbb{Z}_+}$  can be considered as limit cases of the monic Askey-Wilson polynomials by substituting

$$(3.2.1) \quad \underline{t}_L(\varepsilon) := (\varepsilon^{-1}q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, \varepsilon bq^{\frac{1}{2}}, -q^{\frac{1}{2}})$$



for the four variables of the Askey-Wilson polynomials, rescaling of the  $z$ -variable, and taking the limit  $\varepsilon \downarrow 0$ ,

$$(3.2.2) \quad P_n^L(z; a, b) := \lim_{\varepsilon \downarrow 0} \left( \varepsilon q^{-\frac{1}{2}} \right)^n P_n \left( \varepsilon^{-1} q^{\frac{1}{2}} z; \underline{t}_L(\varepsilon) \right)$$

$$(3.2.3) \quad = \lim_{\varepsilon \downarrow 0} \left( \varepsilon q^{-\frac{1}{2}} \right)^n P_n \left( \varepsilon^{-1} q^{\frac{1}{2}} z; t_{L,2}(\varepsilon), t_{L,0}(\varepsilon), t_{L,3}(\varepsilon), t_{L,1}(\varepsilon) \right)$$

$$(3.2.4) \quad = \frac{(qb; q)_n}{(qb)^n (q^{n+1}ab; q)_n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, q^{n+1}ab, qbz \\ qb, 0 \end{matrix}; q, q \right)$$

$$(3.2.5) \quad = \frac{(-1)^n q^{\binom{n}{2}} (qa; q)_n}{(q^{n+1}ab; q)_n} {}_2\phi_1 \left( \begin{matrix} q^{-n}, q^{n+1}ab \\ qa \end{matrix}; q, qz \right)$$

(cf. [66, Proposition 6.3] and take into account that the Askey-Wilson polynomials used in [66] are written as functions of  $(z + z^{-1})/2$  and are normalized differently). In fact, an easy calculation yields that the right hand side of (3.2.3) is equal to

$$(3.2.6) \quad \frac{(qb; q)_n}{(qb)^n (q^{n+1}ab; q)_n} \sum_{m=0}^n \frac{(q^{-n}, q^{n+1}ab; q)_m}{(q, qb; q)_m} (-\varepsilon q^{m+1}b, -\varepsilon q^{m+1}ab; q)_{n-m} \\ \cdot q^m \prod_{i=0}^{m-1} \left( (1 + \varepsilon^2 b^2 q^{2i+1}) - q^{i+1} b \varepsilon q^{-\frac{1}{2}} h_1(\varepsilon^{-1} q^{\frac{1}{2}} z) \right)$$

with  $h_1(z) := z + z^{-1}$ , so (3.2.4) follows from the observation that  $\lim_{\varepsilon \downarrow 0} (u\varepsilon; q)_m = 1$  and

$$(3.2.7) \quad \lim_{u \downarrow 0} u h_1(u^{-1}z) = z.$$

A transformation formula for terminating  ${}_2\phi_1$  series [30, (III.7), p. 241] yields (3.2.5) and shows that the little  $q$ -Jacobi polynomials are also defined for  $b = 0$ . The little  $q$ -Jacobi polynomial  $P_n^L(z; a, b)$  is a monic polynomial of degree  $n$  in the variable  $z$ . So in the limit (3.2.2) we go from a polynomial in  $z + z^{-1}$  to a polynomial in  $z$ . This can be made more transparent as follows. Expand  $P_n$  in powers of  $z + z^{-1}$ ,

$$P_n(z; \underline{t}) = \sum_{r=0}^n c_{n,r}(\underline{t}) h_r(z) \quad (c_{n,n} = 1)$$

with  $h_r(z) := (h_1(z))^r = (z + z^{-1})^r$ . Then (3.2.7) extends to the limit

$$(3.2.8) \quad \lim_{u \downarrow 0} u^r h_r(u^{-1}z) = z^r \quad (r \in \mathbb{N})$$

so by (3.2.6) and (3.2.8) we conclude that

$$P_n^L(z; a, b) = \sum_{r=0}^n c_{n,r}^L(a, b) z^r$$

with

$$(3.2.9) \quad \lim_{\varepsilon \downarrow 0} (\varepsilon q^{-\frac{1}{2}})^{n-r} c_{n,r}(\underline{t}_L(\varepsilon)) = c_{n,r}^L(a, b).$$

The monic big  $q$ -Jacobi polynomials  $\{P_n^B(\cdot; a, b, c, d)\}_{n \in \mathbb{Z}_+}$  may be considered as limit cases of the monic Askey-Wilson polynomials by substituting

$$(3.2.10) \quad \underline{t}_B(\varepsilon) := (\varepsilon^{-1}(qc/d)^{\frac{1}{2}}, -\varepsilon^{-1}(qd/c)^{\frac{1}{2}}, \varepsilon a(qd/c)^{\frac{1}{2}}, -\varepsilon b(qc/d)^{\frac{1}{2}})$$

for the four variables of the Askey-Wilson polynomials, rescaling of the  $z$ -variable, and taking the limit  $\varepsilon \downarrow 0$ :

$$(3.2.11) \quad \begin{aligned} P_n^B(z; a, b, c, d) &:= \lim_{\varepsilon \downarrow 0} (\varepsilon(cd/q)^{\frac{1}{2}})^n P_n(\varepsilon^{-1}(q/cd)^{\frac{1}{2}}z; \underline{t}_B(\varepsilon)) \\ &= \lim_{\varepsilon \downarrow 0} (\varepsilon(cd/q)^{\frac{1}{2}})^n P_n(\varepsilon^{-1}(q/cd)^{\frac{1}{2}}z; t_{B,2}(\varepsilon), t_{B,0}(\varepsilon), t_{B,1}(\varepsilon), t_{B,3}(\varepsilon)) \\ &= \frac{(qa, -qad/c; q)_n}{(q^{n+1}ab; q)_n (qa/c)^n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, q^{n+1}ab, qza/c \\ qa, -qad/c \end{matrix}; q, q \right) \end{aligned}$$

(cf. [66, Proposition 6.1]). Observe that  $P_n^B(z; a, b, c, d)$  is a monic polynomial of degree  $n$  in the variable  $z$ .

Similarly as in the little  $q$ -Jacobi case, we have

$$P_n^B(z; a, b, c, d) = \sum_{r=0}^n c_{n,r}^B(a, b, c, d) z^r$$

with

$$(3.2.12) \quad c_{n,r}^B(a, b, c, d) = \lim_{\varepsilon \downarrow 0} (\varepsilon(cd/q)^{\frac{1}{2}})^{n-r} c_{n,r}(\underline{t}_B(\varepsilon)).$$

In the next theorem we give the orthogonality relations and norm evaluations for the monic little  $q$ -Jacobi polynomials with parameters  $(a, b) \in V_L$ , where the parameter domain  $V_L$  is defined as follows.

**DEFINITION 3.2.1.** *Let  $V_L$  be the set of parameters  $(a, b)$  for which  $a \in (0, 1/q)$  and  $b \in (-\infty, 1/q)$ .*

Recall the definition of the Jackson  $q$ -integral and the  $q$ -Gamma function, which were introduced in Section 1.2.

**THEOREM 3.2.2.** ([1, Theorem 9]) *Let  $(a, b) \in V_L$ . Then*

$$\int_0^1 (P_m^L P_n^L)(z; a, b) v_L(z; a, b) d_q z = \delta_{m,n} \mathcal{N}^L(n; a, b),$$

with

$$(3.2.13) \quad v_L(z; a, b) := \frac{(qz; q)_\infty}{(qbz; q)_\infty} z^\alpha \quad (a = q^\alpha).$$

The quadratic norms  $\mathcal{N}^L(n)$  of the monic little  $q$ -Jacobi polynomials are explicitly given by

$$\mathcal{N}^L(n; a, b) = \frac{\Gamma_q(n+1)\Gamma_q(n+1+\alpha)\Gamma_q(n+1+\beta)\Gamma_q(n+1+\alpha+\beta)}{\Gamma_q(2n+1+\alpha+\beta)\Gamma_q(2n+2+\alpha+\beta)} q^{(n+\alpha)n},$$

where  $b = q^\beta$ .

Observe that the weights  $v_L(z; a, b)$  are positive since  $(a, b) \in V_L$ . The little  $q$ -Jacobi polynomials were first observed by Hahn [33]. A detailed discussion of the orthogonality relations and norm evaluations was given by Andrews and Askey [1]. The orthogonality relations and norm evaluations were derived from the  $q$ -binomial formula [1, (3.6)], [30, (II.3), p.236] and the  $q$ -Pfaff-Saalschütz formula [1, (3.7)], [30, (II.12), p.237]. The evaluation of the  $q$ -Jackson integral over the weight function

$$(3.2.14) \quad \int_0^1 v_L(z; a, b) d_q z = \frac{\Gamma_q(\alpha+1)\Gamma_q(\beta+1)}{\Gamma_q(2+\alpha+\beta)} \quad (a = q^\alpha, b = q^\beta)$$

is a well-known  $q$ -analogue of the beta integral, and is equivalent with the  $q$ -binomial formula [30, (II.3), p.236] (see §1.2).

We end this section with the orthogonality relations and norm evaluations for the monic big  $q$ -Jacobi polynomials with parameters  $(a, b, c, d)$  in the following parameter domain.

DEFINITION 3.2.3. Let  $V_B$  be the set of parameters  $(a, b, c, d)$  for which  $c, d > 0$  and  $a \in (-c/dq, 1/q)$ ,  $b \in (-d/cq, 1/q)$  or  $a = cu$ ,  $b = -d\bar{u}$  with  $u \in \mathbb{C} \setminus \mathbb{R}$ .

THEOREM 3.2.4. ([3, Section 3]) Let  $(a, b, c, d) \in V_B$ . Then

$$(3.2.15) \quad \int_{-d}^c (P_m^B P_n^B)(z; a, b, c, d) v_B(z; a, b, c, d) d_q z = \delta_{m,n} \mathcal{N}^B(n; a, b, c, d),$$

with

$$(3.2.16) \quad v_B(z; a, b, c, d) := \frac{(qz/c, -qz/d; q)_\infty}{(qaz/c, -qbz/d; q)_\infty}.$$

The quadratic norms  $\mathcal{N}^B(n)$  of the monic big  $q$ -Jacobi polynomials are explicitly given by

$$\begin{aligned} \mathcal{N}^B(n; a, b, c, d) := & \frac{\Gamma_q(n+1)\Gamma_q(n+1+\alpha)\Gamma_q(n+1+\beta)\Gamma_q(n+1+\alpha+\beta)}{\Gamma_q(2n+1+\alpha+\beta)\Gamma_q(2n+2+\alpha+\beta)} \\ & \frac{(cd)^{n+1} q^{\binom{n}{2}} (-c/d, -d/c; q)_\infty}{(c+d)(-q^{n+1}bc/d, -q^{n+1}ad/c; q)_\infty} \end{aligned}$$

where  $a = q^\alpha$  and  $b = q^\beta$ .

Observe that the weights  $v_B(z; a, b, c, d)$  are positive since  $(a, b, c, d) \in V_B$ . The big  $q$ -Jacobi polynomials were first hinted at by Hahn [33]. A detailed discussion of the orthogonality relations and norm evaluations was given by Andrews and Askey [3].

The orthogonality relations and norm evaluations were derived using the  $q$ -Vandermonde formula [3, (3.29)], [30, (II.6), p.236] and the evaluation of the  $q$ -Jackson integral over the weight function

$$(3.2.17) \quad \int_{-d}^c v_B(z; a, b, c, d) d_q z = \frac{\Gamma_q(1 + \alpha) \Gamma_q(1 + \beta)}{\Gamma_q(2 + \alpha + \beta)} \frac{(-c/d, -d/c; q)_\infty cd}{(-qbc/d, -qad/c; q)_\infty (c + d)} \\ = (1 - q)c \frac{(q, -d/c, -qc/d, q^2 ab; q)_\infty}{(qa, qb, -qbc/d, -qad/c; q)_\infty}$$

where  $a = q^\alpha$ ,  $b = q^\beta$ . The summation formula (3.2.17) is a  $q$ -analogue of the beta integral which first appeared in [3, Theorem 1].

It turns out that the orthogonality relations and norm evaluations for the little and big  $q$ -Jacobi polynomials can be obtained by taking the limit (3.2.2) respectively (3.2.11) in the orthogonality relations for the Askey-Wilson polynomials. If one chooses the continuous part of the orthogonality measure for the Askey-Wilson polynomials supported on the circle  $T = \{z \in \mathbb{C} \mid |z| = 1\}$  using Cauchy's Theorem, one can show by Lebesgue's dominated convergence Theorem that the continuous part of the measure disappears in the limit (3.2.2) respectively (3.2.11). See Section 1.2 for a simplified example of this phenomenon. In case of the little  $q$ -Jacobi polynomials, the discrete part of the orthogonality measure blows up to one infinite discrete series of weights, since there is one parameter in  $\underline{t}_L(\varepsilon)$  (3.2.1) which blows up to infinity in absolute value when  $\varepsilon \rightarrow 0$ . In case of the big  $q$ -Jacobi polynomials, the discrete part of the orthogonality measure polynomials blows up to two infinite discrete series of weights, since there are two parameters in  $\underline{t}_B(\varepsilon)$  (3.2.10) which blows up to infinity in absolute value when  $\varepsilon \rightarrow 0$ . Making this explicit, one obtains rigorous proofs of Theorem 3.2.2 and Theorem 3.2.4 as corollaries of the orthogonality relations and norm evaluations of the Askey-Wilson polynomials (see [122] for details). In Section 3.3 and Section 3.4 this approach is used to generalize Theorem 3.2.2 and Theorem 3.2.4 to the multivariable setting.

### 3.3. Limit transitions to little $q$ -Jacobi polynomials

In this section a multivariable analogue of the limit from Askey-Wilson polynomials to little  $q$ -Jacobi polynomials (cf. (3.2.3)) is considered. We will show that in this limit, the positive partly discrete orthogonality measure (Theorem 2.6.6) for the Koornwinder polynomials tends to an infinite discrete measure. The rigorous proof of this fact will be given in Section 3.7. This discrete measure is used to define the multivariable analogues of the little  $q$ -Jacobi polynomials. Orthogonality relations and quadratic norm evaluations for the multivariable little  $q$ -Jacobi polynomials are then derived by taking limits in the corresponding results for the Koornwinder polynomials.

The orthogonality measure of the multivariable little  $q$ -Jacobi polynomials can be expressed most conveniently in terms of certain multidimensional Jackson integrals, which

we will define now first. For a point  $\xi \in (\mathbb{C}^*)^n$ , the Jackson integral of  $f$  over the set

$$(3.3.1) \quad \langle \xi \rangle_n := \{\xi q^\nu \mid \nu \in P(n)\}$$

where  $\xi q^\nu := (\xi_1 q^{\nu_1}, \dots, \xi_n q^{\nu_n})$  and  $P(n)$  is given by (2.4.4), is defined by

$$(3.3.2) \quad \iint_{\langle \xi \rangle_n} f(z) d_q z := (1-q)^n \sum_{\nu \in P(n)} f(\xi q^\nu) \prod_{i=1}^n \xi_i q^{\nu_i}$$

provided that the multium is absolutely convergent. Note that for special points  $\xi = (\xi_1, \xi_1 \gamma, \dots, \xi_1 \gamma^{n-1}) \in (\mathbb{C}^*)^n$ , the multium (3.3.2) can be expressed as an iterated Jackson integral (1.2.2) by

$$(3.3.3) \quad \iint_{\langle \xi \rangle_n} f(z) d_q z = \int_{z_1=0}^{\xi_1} \int_{z_2=0}^{\gamma z_1} \dots \int_{z_n=0}^{\gamma z_{n-1}} f(z) d_q z_n \dots d_q z_1.$$

Let  $A_{\mathbb{R}}^{\mathfrak{S}}$  be the  $\mathbb{R}$ -algebra of  $\mathfrak{S}$ -invariant polynomials in the variables  $z_1, \dots, z_n$ . An  $\mathbb{R}$ -basis for  $A_{\mathbb{R}}^{\mathfrak{S}}$  is given by the set of monomials  $\{\tilde{m}_\lambda\}_{\lambda \in \Lambda}$ , where  $\tilde{m}_\lambda(z) := \sum_{\mu \in \mathfrak{S}\lambda} z^\mu$ . Define a symmetric bilinear form  $\langle \cdot, \cdot \rangle_{L,t}^{a,b}$  on  $A_{\mathbb{R}}^{\mathfrak{S}}$  for  $t \in (0, 1)$  and  $(a, b) \in V_L$  (cf. Definition 3.2.1) by

$$(3.3.4) \quad \langle f, g \rangle_L := \iint_{\langle \rho_L \rangle_n} f(z) g(z) \Delta^L(z) d_q z, \quad f, g \in A_{\mathbb{R}}^{\mathfrak{S}}$$

where  $\rho_{L,i} := t^{i-1}$  and where the weight function  $\Delta^L(z) = \Delta^L(z; a, b; t)$  is given by

$$(3.3.5) \quad \Delta^L(z) := q^{-2\tau^2 \binom{n}{3}} t^{-(\alpha+1) \binom{n}{2}} \left( \prod_{i=1}^n v_L(z_i) \right) \delta_{qJ}(z), \quad (a = q^\alpha, t = q^\tau)$$

with  $v_L$  (3.2.13) the weight function for the one variable little  $q$ -Jacobi polynomials and with interaction factor  $\delta_{qJ}(z; t)$  given by

$$(3.3.6) \quad \delta_{qJ}(z; t) := \prod_{1 \leq i < j \leq n} |z_i - z_j| |z_i|^{2\tau-1} (qt^{-1} z_j / z_i; q)_{2\tau-1}, \quad (t = q^\tau, \tau > 0).$$

It can be shown that the weights  $\Delta^L(z)$  in the bilinear form  $\langle \cdot, \cdot \rangle_L$  are strict positive for  $z \in \langle \rho_L \rangle_n$  and that  $\langle f, g \rangle_L$ , written out as a multidimensional infinite sum, is absolutely convergent for all  $f, g \in A_{\mathbb{R}}^{\mathfrak{S}}$ . For a detailed proof of this fact, we refer the reader to [116, Section 6].

**DEFINITION 3.3.1.** *Let  $t \in (0, 1)$  and  $(a, b) \in V_L$ . The multivariable little  $q$ -Jacobi polynomials  $\{P_\lambda^L(\cdot; a, b; t)\}_{\lambda \in \Lambda}$  are by definition the unique symmetric polynomials which satisfy*

- (a)  $P_\lambda^L = \tilde{m}_\lambda + \sum_{\mu < \lambda} c_{\lambda, \mu}^L \tilde{m}_\mu$  for certain  $c_{\lambda, \mu}^L = c_{\lambda, \mu}^L(a, b; t) \in \mathbb{R}$ ;
- (b)  $\langle P_\lambda^L, \tilde{m}_\mu \rangle_L = 0$  for  $\mu < \lambda$ .

In the following proposition the link is established between Koornwinder polynomials with positive partly discrete orthogonality measure and the little  $q$ -Jacobi polynomials. We use in the formulation of the proposition the notation  $|\lambda| := \sum_{i=1}^n \lambda_i$  for the length of a partition  $\lambda \in \Lambda$ .

**PROPOSITION 3.3.2.** *Let  $t \in (0, 1)$  and  $(a, b) \in V_L$ . There exists a sequence of positive real numbers  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  which converges to 0, such that*

$$(3.3.7) \quad \lim_{k \rightarrow \infty} \left( \prod_{i=1}^n (-\varepsilon_k^{-1} q t^{i-1}, -\varepsilon_k^{-1} q a t^{i-1}; q)_\infty \right) (\varepsilon_k q^{-\frac{1}{2}})^{|\lambda|+|\mu|} \langle m_\lambda, m_\mu \rangle_{\underline{t}_L(\varepsilon_k), t} \\ = 2^n n! (q; q)_\infty^{-2n} (1-q)^{-n} \langle \tilde{m}_\lambda, \tilde{m}_\mu \rangle_{L, t}^{a, b}$$

for all  $\lambda, \mu \in \Lambda$ , where  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  is given by (2.6.11) and  $\underline{t}_L$  is given by (3.2.1).

The proof of the proposition will be given in Section 3.7. Observe that  $\underline{t}_L(\varepsilon) \in V_K$  for  $\varepsilon \in \mathbb{R}_{>0}$  sufficiently small, so  $\langle \cdot, \cdot \rangle_{\underline{t}_L(\varepsilon), t}$  is well defined and positive definite for  $\varepsilon \in \mathbb{R}_{>0}$  sufficiently small by Lemma 2.6.4 and Corollary 2.6.5.

Proposition 3.3.2 will be used to prove that the multivariable little  $q$ -Jacobi polynomials are limit cases of the Koornwinder polynomials and to establish orthogonality relations and norm evaluations for the little  $q$ -Jacobi polynomials with respect to the scalar product  $\langle \cdot, \cdot \rangle_L$  on  $A_{\mathbb{R}}^{\mathfrak{S}}$ .

The following definition of limit transitions between  $\mathfrak{S}$ -invariant Laurent polynomials will be used. Let  $f(\cdot; u)$  ( $u \in \mathbb{R}^*$ ) and  $f$  be  $\mathfrak{S}$ -invariant Laurent polynomials in  $n$  variables  $z_1, \dots, z_n$ , then we write  $\lim_{u \rightarrow 0} f(\cdot; u) = f$  if  $\lim_{u \rightarrow 0} f(z; u) = f(z)$  for all  $z \in (\mathbb{C}^*)^n$ . Observe that the  $\mathbb{R}$ -algebra of  $\mathfrak{S}$ -invariant Laurent polynomials has as  $\mathbb{R}$ -basis the set of monomials  $\{\tilde{m}_\lambda(z)\}_{\lambda \in \tilde{\Lambda}}$ , where  $\tilde{\Lambda} := \{\lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$  and  $\tilde{m}_\lambda(z) := \sum_{\mu \in \mathfrak{S}\lambda} z^\mu$ . If  $f(\cdot; u) = \sum_{\lambda \in \tilde{\Lambda}} c_\lambda(u) \tilde{m}_\lambda$  ( $u \in \mathbb{R}^*$ ) and  $f = \sum_{\lambda \in \tilde{\Lambda}} c_\lambda \tilde{m}_\lambda$  satisfy the additional condition that  $\{\lambda \in \tilde{\Lambda} \mid c_\lambda(u) \neq 0\}$  is contained in some finite  $u$ -independent subset for  $|u|$  sufficiently small, then  $\lim_{u \rightarrow 0} f(\cdot; u) = f$  iff  $\lim_{u \rightarrow 0} c_\lambda(u) = c_\lambda$  for all  $\lambda \in \tilde{\Lambda}$ . Crucial in the limit from Koornwinder polynomials to little  $q$ -Jacobi polynomials is a limit from rescaled monomials  $m_\lambda(z|u)$  to  $\tilde{m}_\lambda(z)$ , where the rescaled monomial  $m_\lambda(z|u)$  for  $u \in \mathbb{R}^*$  is the  $\mathfrak{S}$ -invariant Laurent polynomial given by

$$(3.3.8) \quad m_\lambda(z|u) := u^{|\lambda|} m_\lambda(u^{-1}z), \quad \lambda \in \Lambda,$$

where  $u^{-1}z := (u^{-1}z_1, \dots, u^{-1}z_n)$ . In terms of the basis  $\{\tilde{m}_\mu\}_{\mu \in \tilde{\Lambda}}$ , we have

$$m_\lambda(z|u) = \sum_{\mu \in \tilde{\Lambda} \cap \mathcal{W}\lambda} d_{\lambda, \mu}(u) \tilde{m}_\mu(z), \quad \lambda \in \Lambda$$

with  $d_{\lambda, \mu}(u)$  homogeneous of degree  $|\lambda| - |\mu|$  and  $d_{\lambda, \lambda}(u) \equiv 1$ . Furthermore,  $|\mu| \leq |\lambda|$  if  $\mu \in \mathcal{W}\lambda$  and  $|\lambda| = |\mu|$  iff  $\mu \in \mathfrak{S}\lambda$ . Hence we obtain the limit transitions

$$(3.3.9) \quad \lim_{u \rightarrow 0} m_\lambda(z|u) = \tilde{m}_\lambda(z) \quad (\lambda \in \Lambda).$$

This limit will play a fundamental role for the multivariable generalization of the limit from Askey-Wilson polynomials to little  $q$ -Jacobi polynomials.

The quadratic norm of the little  $q$ -Jacobi polynomials can be expressed in terms of functions  $\mathcal{N}_{qJ}^+(\lambda) = \mathcal{N}_{qJ}^+(\lambda; a, b; t)$  and  $\mathcal{N}_{qJ}^-(\lambda) = \mathcal{N}_{qJ}^-(\lambda; a, b; t)$  which are defined by

$$(3.3.10) \quad \mathcal{N}_{qJ}^+(\lambda) := \prod_{i=1}^n \frac{\Gamma_q(\lambda_i + 1 + (n-i)\tau + \alpha + \beta) \Gamma_q(\lambda_i + 1 + (n-i)\tau + \alpha)}{\Gamma_q(2\lambda_i + 1 + 2(n-i)\tau + \alpha + \beta)} \cdot \prod_{1 \leq j < k \leq n} \left( \frac{\Gamma_q(\lambda_j + \lambda_k + 1 + (2n-j-k+1)\tau + \alpha + \beta)}{\Gamma_q(\lambda_j + \lambda_k + 1 + (2n-j-k)\tau + \alpha + \beta)} \cdot \frac{\Gamma_q(\lambda_j - \lambda_k + (k-j+1)\tau)}{\Gamma_q(\lambda_j - \lambda_k + (k-j)\tau)} \right),$$

$$(3.3.11) \quad \mathcal{N}_{qJ}^-(\lambda) := \prod_{i=1}^n \frac{\Gamma_q(\lambda_i + 1 + (n-i)\tau) \Gamma_q(\lambda_i + 1 + (n-i)\tau + \beta)}{\Gamma_q(2\lambda_i + 2 + 2(n-i)\tau + \alpha + \beta)} \cdot \prod_{1 \leq j < k \leq n} \left( \frac{\Gamma_q(\lambda_j + \lambda_k + 2 + (2n-j-k-1)\tau + \alpha + \beta)}{\Gamma_q(\lambda_j + \lambda_k + 2 + (2n-j-k)\tau + \alpha + \beta)} \cdot \frac{\Gamma_q(\lambda_j - \lambda_k + 1 + (k-j-1)\tau)}{\Gamma_q(\lambda_j - \lambda_k + 1 + (k-j)\tau)} \right),$$

where  $a = q^\alpha$ ,  $b = q^\beta$ ,  $t = q^\tau$  and where  $\Gamma_q(z)$  is the  $q$ -Gamma function given by (1.2.4). For  $\lambda \in \Lambda$ ,  $(a, b) \in V_L$  and  $t \in (0, 1)$  define  $\mathcal{N}^L(\lambda) = \mathcal{N}^L(\lambda; a, b; t)$  by

$$(3.3.12) \quad \mathcal{N}^L(\lambda) := q^{\sum_{i=1}^n (\lambda_i + \alpha + 2(n-i)\tau) \lambda_i} \mathcal{N}_{qJ}^+(\lambda) \mathcal{N}_{qJ}^-(\lambda).$$

Observe that  $\mathcal{N}^L(\lambda)$  is well defined and positive. We have now the following multivariable generalization of Theorem 3.2.2 and of the limit transition (3.2.3).

**THEOREM 3.3.3.** *Let  $(a, b) \in V_L$  and  $t \in (0, 1)$ . There exists a sequence of positive real numbers  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  which converges to 0, such that*

$$(3.3.13) \quad \lim_{k \rightarrow \infty} \left( q^{-\frac{1}{2}\varepsilon_k} \right)^{|\lambda|} P_\lambda \left( q^{\frac{1}{2}\varepsilon_k^{-1}} z; \underline{t}_L(\varepsilon_k); t \right) = P_\lambda^L(z; a, b; t)$$

for all  $\lambda \in \Lambda$ . Furthermore, the polynomials  $\{P_\lambda^L\}_{\lambda \in \Lambda}$  are orthogonal with respect to  $\langle \cdot, \cdot \rangle_L$  and the quadratic norms of the little  $q$ -Jacobi polynomials are given by

$$(3.3.14) \quad \langle P_\lambda^L, P_\lambda^L \rangle_L = \mathcal{N}^L(\lambda), \quad \lambda \in \Lambda.$$

**PROOF.** We write

$$(3.3.15) \quad \begin{aligned} (q^{-\frac{1}{2}\varepsilon})^{|\lambda|} P_\lambda(z; \underline{t}_L(\varepsilon); t) &= \sum_{\mu \leq \lambda} c_{\lambda, \mu}(\varepsilon) (q^{-\frac{1}{2}\varepsilon})^{|\mu|} m_\mu(z), \\ P_\lambda^L(z; a, b; t) &= \sum_{\mu \leq \lambda} c_{\lambda, \mu}^L \tilde{m}_\mu(z) \end{aligned}$$

for the expansions of the Koornwinder polynomial and the multivariable little  $q$ -Jacobi polynomial in terms of monomials. In particular, we have  $c_{\lambda,\lambda}(\varepsilon) = c_{\lambda,\lambda}^L = 1$ .

Let  $\preceq$  be a total order on  $\Lambda$  such that  $\mu \preceq \lambda$  if  $\mu \leq \lambda$ . Let  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  be a sequence in  $\mathbb{R}_{>0}$  converging to 0 such that (3.3.7) is satisfied for all  $\lambda, \mu \in \Lambda$ . We prove that

$$(3.3.16) \quad \lim_{k \rightarrow \infty} c_{\lambda,\mu}(\varepsilon_k) = c_{\lambda,\mu}^L, \quad \forall \mu \leq \lambda,$$

and we prove full orthogonality for the subset  $\{P_\mu^L\}_{\mu \preceq \lambda}$  of multivariable little  $q$ -Jacobi polynomials by induction on  $\lambda \in \Lambda$  along  $\preceq$ . The limit (3.3.13) follows then from (3.3.9), (3.3.15) and (3.3.16), and the quadratic norm evaluations (3.3.14) are then immediate consequences of the quadratic norm evaluations of the Koornwinder polynomials (see Theorem 2.6.6), Proposition 3.3.2, (3.3.16) and the observation that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \left( \prod_{i=1}^n (-\varepsilon^{-1} q t^{i-1}, -\varepsilon^{-1} q a t^{i-1}; q)_\infty \right) (q^{-1/2} \varepsilon)^{2|\lambda|} \mathcal{N}(\lambda; \underline{t}_L(\varepsilon); t) \\ = 2^n n! (q; q)_\infty^{-2n} (1-q)^{-n} \mathcal{N}^L(\lambda; a, b; t). \end{aligned}$$

So it remains to prove the induction step (the case  $\lambda = 0$  being trivial). For  $\lambda \neq 0$ , note that  $\underline{t}_L(\varepsilon) \in V_K$  for  $\varepsilon > 0$  sufficiently small, hence by Theorem 2.6.6 we can write

$$(3.3.17) \quad \begin{aligned} (q^{-\frac{1}{2}} \varepsilon)^{|\lambda|} P_\lambda(z; \underline{t}_L(\varepsilon); t) &= (q^{-\frac{1}{2}} \varepsilon)^{|\lambda|} m_\lambda(z) \\ &\quad - \sum_{\nu < \lambda} d_{\lambda,\nu}(\varepsilon) (q^{-\frac{1}{2}} \varepsilon)^{|\nu|} P_\nu(z; \underline{t}_L(\varepsilon); t) \end{aligned}$$

with

$$d_{\lambda,\nu}(\varepsilon) := \frac{(q^{-\frac{1}{2}} \varepsilon)^{|\lambda|+|\nu|} \langle m_\lambda, P_\nu(\cdot; \underline{t}_L(\varepsilon); t) \rangle_{\underline{t}_L(\varepsilon), t}}{(q^{-\frac{1}{2}} \varepsilon)^{2|\nu|} \langle P_\nu(\cdot; \underline{t}_L(\varepsilon); t), P_\nu(\cdot; \underline{t}_L(\varepsilon); t) \rangle_{\underline{t}_L(\varepsilon), t}}$$

for  $\varepsilon > 0$  sufficiently small. By the induction hypotheses, we also have

$$(3.3.18) \quad P_\lambda^L(z; a, b; t) = \tilde{m}_\lambda(z) - \sum_{\nu < \lambda} d_{\lambda,\nu}^L P_\nu^L(z; a, b; t),$$

with

$$d_{\lambda,\nu}^L = \frac{\langle \tilde{m}_\lambda, P_\nu^L(\cdot; a, b; t) \rangle_{L,t}^{a,b}}{\langle P_\nu^L(\cdot; a, b; t), P_\nu^L(\cdot; a, b; t) \rangle_{L,t}^{a,b}}.$$

It follows that for  $\mu < \lambda$ ,

$$(3.3.19) \quad c_{\lambda,\mu}(\varepsilon) = - \sum_{\mu \leq \nu < \lambda} d_{\lambda,\nu}(\varepsilon) c_{\nu,\mu}(\varepsilon), \quad c_{\lambda,\mu}^L = - \sum_{\mu \leq \nu < \lambda} d_{\lambda,\nu}^L c_{\nu,\mu}^L$$

for  $\varepsilon \in \mathbb{R}_{>0}$  sufficiently small. Again by the induction hypotheses and by Proposition 3.3.2, we obtain

$$(3.3.20) \quad \lim_{k \rightarrow \infty} d_{\lambda,\nu}(\varepsilon_k) = d_{\lambda,\nu}^L \quad \forall \nu < \lambda.$$



So the limits (3.3.16) follow from the induction hypotheses, (3.3.19) and (3.3.20). The orthogonality relations for  $\{P_\mu^L\}_{\mu \leq \lambda}$  now follow by taking limits in the orthogonality relations for the Koornwinder polynomials (see Theorem 2.6.6). This completes the proof of the induction step.  $\square$

Write  $e_r(z) = m_{(1^r)}(z)$  and  $\tilde{e}_r(z) = \tilde{m}_{(1^r)}(z)$  for the  $\mathcal{W}$ -invariant and  $\mathfrak{S}$ -invariant monomials corresponding to the fundamental weights  $(1^r) \in \Lambda$  ( $r \in [1, n]$ ). The monomials  $\{e_r\}_{r=1}^n$  and  $\{\tilde{e}_r\}_{r=1}^n$  are algebraically independent generators of the algebras  $A^{\mathcal{W}}$  and  $A^{\mathfrak{S}}$  respectively. Let  $\hat{P}_\lambda^L$  ( $\lambda \in \Lambda$ ) be the unique polynomial in  $n$  variables  $y = (y_1, \dots, y_n)$  satisfying

$$\hat{P}_\lambda^L(\tilde{e}_1(z), \dots, \tilde{e}_n(z)) = P_\lambda^L(z).$$

Similarly, we set  $\hat{P}_\lambda$  for the unique polynomial in the  $n$  variables  $y$  satisfying

$$\hat{P}_\lambda(e_1(z), \dots, e_n(z)) = P_\lambda(z).$$

The limit transition (3.3.13) can now be reformulated as follows.

**COROLLARY 3.3.4.** *Let  $\lambda \in \Lambda$ ,  $(a, b) \in V_L$  and  $t \in (0, 1)$ , then*

$$\lim_{k \rightarrow \infty} (s_{\varepsilon_k})^{-|\lambda|} \hat{P}_\lambda(s_{\varepsilon_k} y_1, \dots, (s_{\varepsilon_k})^n y_n; \underline{t}_L(\varepsilon_k); t) = \hat{P}_\lambda^L(y; a, b; t)$$

for certain sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to zero, where  $s_\varepsilon := q^{\frac{1}{2}} \varepsilon^{-1}$ .

**PROOF.** The corollary follows from the proof of Theorem 3.3.3 since

$$(s_\varepsilon)^{-r} e_r(s_\varepsilon z) = \tilde{e}_r(z) + \mathcal{O}(\varepsilon)$$

for  $r \in [1, n]$ .  $\square$

Let  $D_L = D_{L,a,b,t}$  be the second order  $q$ -difference operator

$$(3.3.21) \quad D_L = \sum_{j=1}^n (\phi_{L,j}^+(z)(T_j^+ - \text{Id}) + \phi_{L,j}^-(z)(T_j^- - \text{Id}))$$

where  $\phi_{L,j}^\pm(z) = \phi_{L,j}^\pm(z; a, b; t)$  is given by

$$(3.3.22) \quad \begin{aligned} \phi_{L,j}^+(z) &:= qt^{n-1} a \left( b - \frac{1}{qz_j} \right) \prod_{i \neq j} \frac{z_i - tz_j}{z_i - z_j}, \\ \phi_{L,j}^-(z) &:= \left( 1 - \frac{1}{z_j} \right) \prod_{i \neq j} \frac{z_j - tz_i}{z_j - z_i}. \end{aligned}$$

Then we have the following corollary of Theorem 3.3.3.

**COROLLARY 3.3.5.** *Let  $(a, b) \in V_L$  and  $t \in (0, 1)$ . Then  $P_\lambda^L(\cdot; a, b; t)$  is an eigenfunction of  $D_{L,a,b,t}$  with eigenvalue*

$$(3.3.23) \quad E_\lambda^{qJ}(a, b; t) := \sum_{j=1}^n (qabt^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1))$$

for all  $\lambda \in \Lambda$ .

PROOF. Follows from (3.3.13) by taking the limit  $k \rightarrow \infty$  in the equations

$$(3.3.24) \quad (q^{-1}\varepsilon_k)^{|\lambda|} ((D - E_\lambda)P_\lambda)(q^{\frac{1}{2}}\varepsilon_k^{-1}z; \underline{t}_L(\varepsilon_k); t) = 0 \quad (\lambda \in \Lambda)$$

where  $D$  is given by (2.3.13) and  $E_\lambda$  is given by (2.3.16).  $\square$

In [16] van Diejen introduced  $n$  commuting  $q$ -difference operators  $D_i$  ( $i \in [1, n]$ ) with  $D_1 = D$  which simultaneously diagonalize the Koornwinder polynomials. It is possible to explicitly compute the limit (3.3.13) in the corresponding eigenvalue equations, as was done for  $D_1 = D$  in Corollary 3.3.5. This yields  $n$  explicit, commuting  $q$ -difference operators  $D_{L,1} = D_L, D_{L,2}, \dots, D_{L,n}$  which simultaneously diagonalize the little  $q$ -Jacobi polynomials. The explicit formulas are omitted here, since we will not be needing them in the remainder of the thesis.

The constant term identity for the little  $q$ -Jacobi polynomials can be rewritten as follows.

COROLLARY 3.3.6. For  $t \in (0, 1)$  and  $(a, b) \in V_L$  we have

$$(3.3.25) \quad \langle 1, 1 \rangle_{L,t}^{a,b} = \prod_{j=1}^n \frac{\Gamma_q(\alpha + 1 + (j-1)\tau)\Gamma_q(\beta + 1 + (j-1)\tau)\Gamma_q(j\tau)}{\Gamma_q(\alpha + \beta + 2 + (n+j-2)\tau)\Gamma_q(\tau)},$$

where  $a = q^\alpha, b = q^\beta$  and  $t = q^\tau$ .

The constant term identity (3.3.25) has been studied extensively in the past 20 years. It was conjectured by Askey [5] for  $t = q^k, k \in \mathbb{N}$  and proved in this case independently by Habsieger [33] and Kadell [54] (see Section 3.5 for more details). For arbitrary  $t \in (0, 1)$  the first proof appeared in Aomoto's paper [4] (see also [55] and [127] for alternative proofs).

### 3.4. Limit transitions to big $q$ -Jacobi polynomials

In this section a multivariable analogue of the limit transition from Askey-Wilson polynomials to big  $q$ -Jacobi polynomials (cf. (3.2.11)) is considered. By repeating the methods of the previous section, we derive orthogonality relations and quadratic norm evaluations for multivariable analogues of the big  $q$ -Jacobi polynomials.

Before we define the orthogonality measure for the multivariable big  $q$ -Jacobi polynomials we first need to introduce some more notations. Set

$$\langle \xi, \eta \rangle_n := \bigcup_{j=0}^n \langle \xi \rangle_j \times \langle \eta \rangle_{n-j} \subset \mathbb{C}^n$$

where  $\eta, \xi \in (\mathbb{C}^*)^n$  and  $\langle \xi \rangle_n$  is defined by (3.3.1). Here we use the convention that  $\langle \xi \rangle_n \times \langle \eta \rangle_0 = \langle \xi \rangle_n$  and  $\langle \xi \rangle_0 \times \langle \eta \rangle_n = \langle \eta \rangle_n$ . Let  $\underline{c} = (c_0, \dots, c_n) \in (\mathbb{C}^*)^{n+1}$ , then we

define the  $\underline{c}$ -weighted Jackson integral of  $f$  over the set  $\langle \eta, \xi \rangle_n$  by

$$(3.4.1) \quad \begin{aligned} \iint_{\langle \xi, \eta \rangle_n} f(z) d_q^{\underline{c}} z &:= \sum_{j=0}^n (-1)^{n-j} c_j \iint_{z \in \langle \xi \rangle_j} \iint_{w \in \langle \eta \rangle_{n-j}} f(z, w) d_q z d_q w \\ &= (1-q)^n \sum_{j=0}^n \sum_{\substack{\mu \in P(j) \\ \nu \in P(n-j)}} c_j f(\xi q^\mu, \eta q^\nu) \prod_{l=1}^j \xi_l q^{\mu_l} \prod_{m=1}^{n-j} (-\eta_m q^{\nu_m}), \end{aligned}$$

where the  $j = 0$  respectively  $j = n$  term in (3.4.1) should be read as

$$(-1)^n c_0 \iint_{w \in \langle \eta \rangle_n} f(w) d_q w = (1-q)^n \sum_{\nu \in P(n)} c_0 f(\eta q^\nu) \prod_{m=1}^n (-\eta_m q^{\nu_m})$$

respectively

$$c_n \iint_{z \in \langle \xi \rangle_n} f(z) d_q z = (1-q)^n \sum_{\mu \in P(n)} c_n f(\xi q^\mu) \prod_{l=1}^n \xi_l q^{\mu_l}.$$

If  $\eta = (\eta_1, \eta_1 \gamma', \dots, \eta_1 (\gamma')^{n-1})$  and  $\xi = (\xi_1, \xi_1 \gamma, \dots, \xi_1 \gamma^{n-1})$ , then the  $\underline{c}$ -weighted Jackson integral over  $\langle \eta, \xi \rangle_n$  can be rewritten as an iterated Jackson integral by

$$\begin{aligned} \iint_{\langle \xi, \eta \rangle_n} f(z) d_q^{\underline{c}} z &= \sum_{j=0}^n c_j \int_{z_1=0}^{\xi_1} \int_{z_2=0}^{\gamma z_1} \dots \int_{z_j=0}^{\gamma z_{j-1}} \\ &\quad \int_{z_{j+1}=\eta_1}^0 \int_{z_{j+2}=\gamma' z_{j+1}}^0 \dots \int_{z_n=\gamma' z_{n-1}}^0 f(z) d_q z_n \dots d_q z_1. \end{aligned}$$

Define a symmetric bilinear form  $\langle \cdot, \cdot \rangle_{B,t}^{a,b,c,d}$  on  $A_{\mathbb{R}}^{\otimes}$  for  $t \in (0, 1)$  and  $(a, b, c, d) \in V_B$  (cf. Definition 3.2.3) by

$$(3.4.2) \quad \langle f, g \rangle_B := \iint_{\langle \rho_B, \sigma_B \rangle_n} f(z) g(z) \Delta^B(z) d_q^{\underline{c}_B} z, \quad f, g \in A_{\mathbb{R}}^{\otimes},$$

with  $\rho_{B,i} := ct^{i-1}$ ,  $\sigma_{B,i} := -dt^{i-1}$  and with weight function

$$(3.4.3) \quad \Delta^B(z) := \left( \prod_{i=1}^n v_B(z_i) \right) \delta_{qJ}(z),$$

where  $v_B$  (3.2.16) is the weight function in the orthogonality measure for the one variable big  $q$ -Jacobi polynomials and  $\delta_{qJ}(z) = \delta_{qJ}(z; t)$  is given by (3.3.6). The weight  $\underline{c}_B = \underline{c}_B(c, d; t)$  is of the form  $c_{B,j} := c_B d_{B,j}$ , with

$$(3.4.4) \quad d_{B,j} := \prod_{\substack{1 \leq k < m \leq n \\ k \leq j}} \Psi_t(-t^{n-m-k+1} d/c)$$

where  $\Psi_t(x)$  is defined by

$$(3.4.5) \quad \Psi_t(x) := |x|^{2\tau-1} \frac{\theta(tx)}{\theta(qt^{-1}x)}$$

with  $\theta(x)$  the Jacobi theta function

$$(3.4.6) \quad \theta(x) := (q, x, qx^{-1}; q)_\infty.$$

The constant  $c_B$  is defined by

$$(3.4.7) \quad c_B := \frac{(q; q)_\infty^n q^{-2\tau^2 \binom{n}{3}} d^{-2\tau \binom{n}{2} - n} t^{-\binom{n}{2}}}{\prod_{i=1}^n \theta(-t^{1-i} c/d)}.$$

The positive constant  $c_B$  is not essential for the definition of  $\langle \cdot, \cdot \rangle_B$ . We have chosen to take this constant within the definition of  $\langle \cdot, \cdot \rangle_B$  because it will simplify formulas and notations later on. The Jacobi theta function satisfies the functional relation

$$(3.4.8) \quad \theta(q^k x) = (-x^{-1})^k q^{-\binom{k}{2}} \theta(x), \quad k \in \mathbb{Z}_+.$$

This implies that  $\Psi_t$  is a quasi constant, i.e.  $\Psi_t(qx) = \Psi_t(x)$ . In particular, the weight  $\underline{d}_B$  (3.4.4) is independent of  $a, b$  and quasi constant in the parameters  $c, d$ .

The weight  $\underline{c}_B$  in the definition of  $\langle \cdot, \cdot \rangle_B$  is needed in order to obtain good asymptotic behaviour of the weights occurring in  $\langle \cdot, \cdot \rangle_B$ . To be more precise, let  $j \in [1, n]$ ,  $\lambda \in P(j-1)$ ,  $\mu \in P(n-j)$  and set  $\lambda^{(l)} := (\lambda, l) \in P(j)$  for  $l \geq \lambda_{j-1}$ , respectively  $\mu^{(m)} := (\mu, m) \in P(n-j+1)$  for  $m \geq \mu_{n-j}$ . For  $j=1$  respectively  $j=n$ , this should be read as  $\lambda^{(l)} = l \in P(1)$ , respectively  $\mu^{(m)} = m \in P(1)$ . Define

$$(3.4.9) \quad \begin{aligned} z^+(l; \lambda, \mu) &:= (\rho_B q^{\lambda^{(l)}}, \sigma_B q^\mu) \in \langle \rho_B \rangle_j \times \langle \sigma_B \rangle_{n-j} \quad (l \geq \lambda_{j-1}), \\ z^-(m; \lambda, \mu) &:= (\rho_B q^\lambda, \sigma_B q^{\mu^{(m)}}) \in \langle \rho_B \rangle_{j-1} \times \langle \sigma_B \rangle_{n-j+1} \quad (m \geq \mu_{n-j}), \end{aligned}$$

then we have

$$\lim_{l \rightarrow \infty} z^+(l; \lambda, \mu) = (\rho_B q^\lambda, 0, \sigma_B q^\mu), \quad \lim_{m \rightarrow \infty} z^-(m; \lambda, \mu) = (\rho_B q^\lambda, \sigma_B q^\mu, 0)$$

(with the obvious conventions when  $j=1$  respectively  $j=n$ ). We have now good asymptotic behaviour of the weights in the following sense.

LEMMA 3.4.1. *Let  $(a, b, c, d) \in V_B$  and  $t \in (0, 1)$ . Then*

$$(3.4.10) \quad \lim_{l \rightarrow \infty} c_{B,j} \Delta^B(z^+(l; \lambda, \mu)) = \lim_{m \rightarrow \infty} c_{B,j-1} \Delta^B(z^-(m; \lambda, \mu))$$

for all  $\lambda \in P(j-1)$ ,  $\mu \in P(n-j)$  and  $j \in [1, n]$ . The conditions (3.4.10) for  $\lambda \in P(j-1)$ ,  $\mu \in P(n-j)$  and  $j \in [1, n]$  characterize the weight  $\underline{c}_B$  uniquely up to a multiplicative constant.

PROOF. Let  $\lambda \in P(j-1)$  and  $\mu \in P(n-j)$ . By direct calculation we obtain

$$\lim_{m \rightarrow \infty} \Delta^B(z^-(m; \lambda, \mu)) = \delta_{qJ} (\rho_B q^\lambda, \sigma_B q^\mu) \prod_{i=1}^{j-1} (ct^{i-1} q^{\lambda_i})^{2\tau} \prod_{k=1}^{n-j} (dt^{k-1} q^{\mu_k})^{2\tau},$$

where  $t = q^\tau$  and  $\delta_{qJ}$  is the interaction factor (3.3.6) considered as function of  $n - 1$  variables. Using the equality

$$q^{k(2\tau-1)}(q^{1-\tau-k}x; q)_{2\tau-1} = \frac{(q^\tau x^{-1}; q)_k}{(q^{1-\tau}x^{-1}; q)_k} (q^{1-\tau}x; q)_{2\tau-1}$$

for  $k \in \mathbb{Z}_+$  we furthermore have that

$$\begin{aligned} \lim_{l \rightarrow \infty} \Delta^B(z^+(l; \lambda, \mu)) &= \delta_{qJ}(\rho_B q^\lambda, \sigma_B q^\mu) \\ &\cdot \prod_{i=1}^{j-1} (ct^{i-1} q^{\lambda_i})^{2\tau} \prod_{k=1}^{n-j} \left\{ (dt^{k-1} q^{\mu_k})^{2\tau} \Psi_t(-t^{j-k} q^{-\mu_k} c/d) \right\}. \end{aligned}$$

The lemma follows now from the formula

$$\frac{c_{B,j-1}}{c_{B,j}} = \prod_{k=1}^{n-j} \Psi_t(-t^{j-k} c/d)$$

since  $\Psi_t$  is a quasi-constant.  $\square$

Finally, it can be shown that the bilinear form  $\langle \cdot, \cdot \rangle_B$  (3.4.2) on  $A_{\mathbb{R}}^{\mathfrak{S}}$  is well defined and positive definite. Indeed, for parameters  $(a, b, c, d) \in V_B$  and  $t \in (0, 1)$  it can be shown that the weights  $\Delta^B(z)$  in the bilinear form  $\langle \cdot, \cdot \rangle_B$  are strict positive for  $z \in \langle \rho_B, \sigma_B \rangle_n$  and that  $\langle f, g \rangle_B$ , written out as a multidimensional infinite sum over  $\langle \sigma_B, \rho_B \rangle_n$ , is absolutely convergent for all  $f, g \in A_{\mathbb{R}}^{\mathfrak{S}}$ . We refer the reader to [116, Section 6] for a detailed proof of this fact.

**DEFINITION 3.4.2.** *Let  $t \in (0, 1)$  and  $(a, b, c, d) \in V_B$ . The multivariable big  $q$ -Jacobi polynomials  $\{P_\lambda^B(\cdot; a, b, c, d; t)\}_{\lambda \in \Lambda}$  are by definition the unique symmetric polynomials which satisfy*

- (a)  $P_\lambda^B = \tilde{m}_\lambda + \sum_{\mu < \lambda} c_{\lambda, \mu}^B \tilde{m}_\mu$  for certain constants  $c_{\lambda, \mu}^B = c_{\lambda, \mu}^B(a, b, c, d; t) \in \mathbb{R}$ ;
- (b)  $\langle P_\lambda^B, \tilde{m}_\mu \rangle_B = 0$  for  $\mu < \lambda$ .

The following lemma shows that the multivariable big  $q$ -Jacobi polynomials depend essentially only on  $a, b, t$  and the ratio  $c/d$ .

**LEMMA 3.4.3.** *For  $t \in (0, 1)$  and  $(a, b, c, d) \in V_B$  we have*

$$(3.4.11) \quad d^{-|\lambda|} P_\lambda^B(dz; a, b, c, d; t) = P_\lambda^B(z; a, b, c/d, 1; t), \quad \forall \lambda \in \Lambda.$$

**PROOF.** For  $f \in A_{\mathbb{R}}^{\mathfrak{S}}$  set  $f_d(z) := f(dz)$ , where  $dz := (dz_1, \dots, dz_n)$ . Then it suffices to show that

$$\langle f, g \rangle_{B,t}^{a,b,c,d} = \langle f_d, g_d \rangle_{B,t}^{a,b,\frac{c}{d},1}, \quad \forall f, g \in A_{\mathbb{R}}^{\mathfrak{S}}.$$

But this is a direct consequence of the formulas

$$(3.4.12) \quad \int_0^{\alpha\gamma} h(x) d_q x = \alpha \int_0^\gamma h(\alpha x) d_q x,$$

$$\underline{c}_B(c, d; t) = d^{-2\tau \binom{n}{2} - n} \underline{c}_B(c/d, 1; t)$$

and  $\Delta^B(dz; a, b, c, d; t) = d^{2\tau \binom{n}{2}} \Delta^B(z; a, b, c/d, 1; t)$ , where  $t = q^\tau$ .  $\square$

The following proposition is the analogue of Proposition 3.3.2 in case of the big  $q$ -Jacobi polynomials.

**PROPOSITION 3.4.4.** *Let  $t \in (0, 1)$  and  $(a, b, c, d) \in V_B$ . There exists a sequence of positive real numbers  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  which converges to 0, such that*

$$(3.4.13) \quad \lim_{k \rightarrow \infty} \left( \prod_{i=1}^n (-\varepsilon_k^{-2} q t^{i-1}; q)_\infty \right) (\varepsilon_k (cd/q)^{\frac{1}{2}})^{|\lambda|+|\mu|} \langle m_\lambda, m_\mu \rangle_{\underline{t}_B(\varepsilon_k), t} \\ = 2^n n! (q; q)_\infty^{-2n} (1-q)^{-n} \langle \tilde{m}_\lambda, \tilde{m}_\mu \rangle_{B, t}^{a, b, c, d}$$

for all  $\lambda, \mu \in \Lambda$ , where  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  is given by (2.6.11) and  $\underline{t}_B$  is given by (3.2.10).

The proof will be given in Section 3.8. Note that  $\underline{t}_B(\varepsilon) \in V_K$  for  $\varepsilon \in \mathbb{R}_{>0}$  sufficiently small, so  $\langle \cdot, \cdot \rangle_{\underline{t}_B(\varepsilon), t}$  is well defined and positive definite for  $\varepsilon \in \mathbb{R}_{>0}$  sufficiently small by Lemma 2.6.4 and Corollary 2.6.5.

The arguments of Section 3.3 can now be repeated to establish full orthogonality of the big  $q$ -Jacobi polynomials with respect to  $\langle \cdot, \cdot \rangle_B$  and to calculate their quadratic norms. For  $\lambda \in \Lambda$ ,  $(a, b, c, d) \in V_B$  and  $t \in (0, 1)$  let  $\mathcal{N}^B(\lambda) = \mathcal{N}^B(\lambda; a, b, c, d; t)$  be given by

$$(3.4.14) \quad \mathcal{N}^B(\lambda) := (cd)^{|\lambda|} q^{\frac{1}{2} \sum_{i=1}^n (\lambda_i - 1 + 2(n-i)\tau) \lambda_i} \mathcal{N}_{qJ}^+(\lambda; a, b; t) \mathcal{N}_{qJ}^-(\lambda; a, b; t) \\ \cdot \prod_{i=1}^n (-q^{\lambda_i+1} b t^{n-i} c/d, -q^{\lambda_i+1} a t^{n-i} d/c; q)_\infty^{-1}$$

where  $t = q^\tau$  and  $\mathcal{N}_{qJ}^+$  respectively  $\mathcal{N}_{qJ}^-$  is given by (3.3.10) respectively (3.3.11). Observe that  $\mathcal{N}^B(\lambda)$  is well defined and positive. We have now the following multivariable analogue of Theorem 3.2.4 and of the limit transition (3.2.11).

**THEOREM 3.4.5.** *Let  $t \in (0, 1)$  and  $(a, b, c, d) \in V_B$ . There exists a sequence of positive real numbers  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  which converges to 0, such that*

$$(3.4.15) \quad \lim_{k \rightarrow \infty} \left( \varepsilon_k (cd/q)^{\frac{1}{2}} \right)^{|\lambda|} P_\lambda \left( (q/cd)^{\frac{1}{2}} \varepsilon_k^{-1} z; \underline{t}_B(\varepsilon_k); t \right) = P_\lambda^B(z; a, b, c, d; t)$$

for all  $\lambda \in \Lambda$ . Furthermore, the polynomials  $\{P_\lambda^B\}_{\lambda \in \Lambda}$  are orthogonal with respect to  $\langle \cdot, \cdot \rangle_B$  and the quadratic norms of the big  $q$ -Jacobi polynomials are given by

$$(3.4.16) \quad \langle P_\lambda^B, P_\lambda^B \rangle_B = \mathcal{N}^B(\lambda), \quad \lambda \in \Lambda.$$

**PROOF.** We have the limit

$$\lim_{\varepsilon \rightarrow 0} \left( \prod_{i=1}^n (-\varepsilon^{-2} q t^{i-1}; q)_\infty \right) ((cd/q)^{\frac{1}{2}} \varepsilon)^{2|\lambda|} \mathcal{N}(\lambda; \underline{t}_B(\varepsilon); t) \\ = 2^n n! (q; q)_\infty^{-2n} (1-q)^{-n} \mathcal{N}^B(\lambda; a, b, c, d; t).$$

The proof is now analogous to the proof of Theorem 3.3.3.  $\square$

The quadratic norm evaluations of the big  $q$ -Jacobi polynomials for the special case  $a = b = 0, c = 1$  and  $t = q^k$  with  $k \in \mathbb{N}$  are recently proved by Baker and Forrester [8, Section 4.3] using Pieri formulas. In order to see that the quadratic norms of the big  $q$ -Jacobi polynomials in this special case are in agreement with the quadratic norm evaluations [8, (4.3)], one needs to use the evaluation formula for the Macdonald polynomials [85, (6.11)] together with [55, Proposition 3.2].

Recall the monomials  $\tilde{e}_r(z) = \tilde{m}_{(1^r)}(z)$  for  $r \in [1, n]$ , which are algebraically independent generators of  $A^\ominus$ . Let  $\hat{P}_\lambda^B$  ( $\lambda \in \Lambda$ ) be the unique polynomial in the  $n$  variables  $y = (y_1, \dots, y_n)$  satisfying

$$\hat{P}_\lambda^B(\tilde{e}_1(z), \dots, \tilde{e}_n(z)) = P_\lambda^B(z).$$

The limit transition (3.4.15) can now be reformulated as follows (cf. Corollary 3.3.4).

**COROLLARY 3.4.6.** *Let  $\lambda \in \Lambda$ ,  $(a, b, c, d) \in V_B$  and  $t \in (0, 1)$ , then*

$$\lim_{k \rightarrow \infty} (s_{\varepsilon_k})^{-|\lambda|} \hat{P}_\lambda(s_{\varepsilon_k} y_1, \dots, (s_{\varepsilon_k})^n y_n; t_B(\varepsilon_k); t) = \hat{P}_\lambda^B(y; a, b, c, d; t)$$

for certain sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to zero, where  $s_\varepsilon := \varepsilon^{-1}(q/cd)^{\frac{1}{2}}$ .

Let  $D_B = D_{B,a,b,c,d,t}$  be the second order  $q$ -difference operator

$$(3.4.17) \quad D_B = \sum_{j=1}^n (\phi_{B,j}^+(z)(T_j^+ - \text{Id}) + \phi_{B,j}^-(z)(T_j^- - \text{Id}))$$

where  $\phi_{B,j}^\pm(z) = \phi_{B,j}^\pm(z; a, b, c, d; t)$  is given by

$$(3.4.18) \quad \begin{aligned} \phi_{B,j}^+(z) &:= qt^{n-1} \left( a - \frac{c}{qz_j} \right) \left( b + \frac{d}{qz_j} \right) \prod_{i \neq j} \frac{z_i - tz_j}{z_i - z_j}, \\ \phi_{B,j}^-(z) &:= \left( 1 - \frac{c}{z_j} \right) \left( 1 + \frac{d}{z_j} \right) \prod_{i \neq j} \frac{z_j - tz_i}{z_j - z_i}. \end{aligned}$$

Then we have the following corollary of Theorem 3.3.3.

**COROLLARY 3.4.7.** *Let  $(a, b, c, d) \in V_B$  and  $t \in (0, 1)$ . Then  $P_\lambda^B(\cdot; a, b, c, d; t)$  is an eigenfunction of  $D_{B,a,b,c,d,t}$  with eigenvalue  $E_\lambda^{qJ}(a, b; t)$  (3.3.23) for all  $\lambda \in \Lambda$ .*

**PROOF.** The proof is analogous to the proof of Corollary 3.3.5.  $\square$

It follows from Theorem 3.4.5 that  $D_B$  is symmetric with respect to  $\langle \cdot, \cdot \rangle_B$ . In [116] the symmetry of  $D_B$  was established by direct calculations in which the asymptotic behaviour of the weight function  $\Delta^B$  (see Lemma 3.4.1) plays a crucial role. Similarly as in the little  $q$ -Jacobi polynomial case (see after Corollary 3.3.5), limits can be taken of the  $n$  commuting  $q$ -difference operators  $D_1 := D, D_2, \dots, D_n$  of van Diejen [16], yielding  $n$  commuting  $q$ -difference operators  $D_{B,1} = D_B, D_{B,2}, \dots, D_{B,n}$  which simultaneously diagonalize the big  $q$ -Jacobi polynomials. The explicit form of the  $q$ -difference operators

$D_{B,i}$  and their eigenvalues can be computed directly (see Corollary 3.4.7 for the special case  $D_{B,1} = D_B$ ).

The constant term identity for the big  $q$ -Jacobi polynomials can be rewritten as follows.

**COROLLARY 3.4.8.** *Let  $t \in (0, 1)$  and  $(a, b, c, d) \in V_B$ . Then*

$$(3.4.19) \quad \langle 1, 1 \rangle_{B,t}^{a,b,c,d} = \prod_{j=1}^n \left( \frac{\Gamma_q(\alpha + 1 + (j-1)\tau) \Gamma_q(\beta + 1 + (j-1)\tau) \Gamma_q(j\tau)}{\Gamma_q(\alpha + \beta + 2 + (n+j-2)\tau) \Gamma_q(\tau)} \right) \cdot (-q^{\alpha+1+(j-1)\tau} d/c, -q^{\beta+1+(j-1)\tau} c/d; q)_{\infty}^{-1}$$

where  $a = q^\alpha$ ,  $b = q^\beta$  and  $t = q^\tau$ .

The constant term identity (3.4.19) was conjectured by Askey [5] for  $t = q^k$ ,  $k \in \mathbb{N}$ , and proved in this case by Evans [29]. See the next section for more details on the special case  $t = q^k$ ,  $k \in \mathbb{N}$ . The evaluation (3.4.19) for general  $t \in (0, 1)$  is equivalent to Tarasov's and Varchenko's summation formula [127, Theorem (E.10)]. The proof of Tarasov and Varchenko is by computing residues for an A type generalization of Askey-Roy's  $q$ -beta integral.

### 3.5. The special case $t = q^k$ with $k \in \mathbb{N}$

In this section the results of the previous sections are considered for deformation parameter  $t = q^k$  with  $k \in \mathbb{N}$ . Many formulas simplify under this assumption because  $t$  is "compatible" with the base  $q$  when  $t = q^k$  with  $k \in \mathbb{N}$ . For instance  $q$ -shifted factorials  $(a; q)_k$ , which for general  $k > 0$  are given by quotients of infinite products, simplify to a finite product when  $k \in \mathbb{N}$ . In the remainder of this section we assume that  $k \in \mathbb{N}$  is fixed.

Recall that in the definition of the orthogonality measure for the big  $q$ -Jacobi polynomials the so-called  $\mathcal{C}$ -weighted Jackson integral is used, where the weight for the big  $q$ -Jacobi polynomials is given by  $c_{B,j} = c_B d_{B,j}$  ( $j \in [0, n]$ ) with  $d_{B,j}$  given by (3.4.4) and  $c_B$  given by (3.4.7). For  $t = q^k$  it is easily verified that  $d_{B,j} = 1$  for all  $j \in [0, n]$ . For the weight  $\mathcal{C}_B$  we thus have  $c_{B,j} = c_B$  for  $t = q^k$  independent of  $j$ , and  $c_B$  can then be rewritten as

$$(3.5.1) \quad c_B = \frac{q^{\binom{k}{2} \binom{n}{2} - k^2 \binom{n}{3}} (c+d)^n}{(-d/c, -c/d; q)_{\infty}^n (cd)^{n + \binom{n}{2} k}} \quad (t = q^k).$$

This follows by a straightforward calculation using the relation  $\theta(qx^{-1}) = \theta(x)$ , (3.4.8) and

$$(3.5.2) \quad \sum_{i=1}^n (i-1)^2 = 2 \binom{n}{3} + \binom{n}{2}.$$



Now set

$$(3.5.3) \quad \begin{aligned} \tilde{\delta}_{qJ}(z; q^k) &:= \prod_{1 \leq i < j \leq n} (z_i - z_j) z_i^{2k-1} (q^{1-k} z_j / z_i; q)_{2k-1} \\ &= (-1)^k \binom{n}{2} q^{-\binom{k}{2} \binom{n}{2}} \prod_{\substack{i \neq j \\ l \in [0, k-1]}} (z_i - q^l z_j) \end{aligned}$$

and write  $\tilde{\Delta}^B(\cdot; q^k)$  respectively  $\tilde{\Delta}^L(\cdot; q^k)$  for the weight function (3.4.3) respectively (3.3.5) with the interaction factor  $\delta_{qJ}(\cdot; q^k)$  (3.3.6) replaced by  $\tilde{\delta}_{qJ}(\cdot; q^k)$  (3.5.3).

LEMMA 3.5.1. *Let  $t = q^k$  ( $k \in \mathbb{N}$ ) and fix  $(a, b) \in V_L$  respectively  $(a, b, c, d) \in V_B$ . Then*

$$(3.5.4) \quad \langle f, g \rangle_{L, q^k} = \frac{1}{n!} \int_{z_1=0}^1 \int_{z_2=0}^1 \cdots \int_{z_n=0}^1 f(z) g(z) \tilde{\Delta}^L(z; q^k) d_q z$$

respectively

$$(3.5.5) \quad \langle f, g \rangle_{B, q^k} = \frac{c_B}{n!} \int_{z_1=-d}^c \int_{z_2=-d}^c \cdots \int_{z_n=-d}^c f(z) g(z) \tilde{\Delta}^B(z; q^k) d_q z$$

for all  $f, g \in A_{\mathbb{R}}^{\otimes}$ , where  $d_q z = d_q z_n \cdots d_q z_2 d_q z_1$ .

PROOF. We give a proof for the big  $q$ -Jacobi orthogonality measure. Observe that  $\delta_{qJ}(z; q^k) = 0$  if  $z_i = q^l z_j$  for certain  $i \neq j$  and  $l \in [0, k-1]$  (and similarly for  $\tilde{\delta}_{qJ}$ ). Furthermore, we have already seen that the weight  $c_{B,j}$  is independent of  $j \in [0, n]$ , and that it is equal to  $c_B$ . It follows that  $c_B^{-1} \langle f, g \rangle_{B, q^k}$  for  $f, g \in A_{\mathbb{R}}^{\otimes}$  can be rewritten as

$$(3.5.6) \quad \sum_{j=0}^n \int_{z_1=0}^c \int_{z_2=0}^{z_1} \cdots \int_{z_j=0}^{z_{j-1}} \int_{z_{j+1}=-d}^0 \int_{z_{j+2}=z_{j+1}}^0 \cdots \int_{z_n=z_{n-1}}^0 f(z) g(z) \Delta^B(z) d_q z.$$

For a mass point  $z = (z_1, \dots, z_j, z_{j+1}, \dots, z_n)$  occurring in the  $j$ th term of (3.5.6), it is easily verified that

$$\Delta^B(z; q^k) = \tilde{\Delta}^B(z_1, \dots, z_j, z_n, z_{n-1}, \dots, z_{n-j+1}; q^k),$$

hence (3.5.6) can be rewritten as

$$\sum_{j=0}^n \int_{z_1=0}^c \int_{z_2=0}^{z_1} \cdots \int_{z_j=0}^{z_{j-1}} \int_{z_{j+1}=-d}^0 \int_{z_{j+2}=-d}^{z_{j+1}} \cdots \int_{z_n=-d}^{z_{n-1}} f(z) g(z) \tilde{\Delta}^B(z) d_q z,$$

which in turn is equal to

$$(3.5.7) \quad \int_{z_1=-d}^c \int_{z_2=-d}^{z_1} \cdots \int_{z_n=-d}^{z_{n-1}} f(z) g(z) \tilde{\Delta}^B(z) d_q z.$$

Now  $\tilde{\Delta}^B(z; q^k)$  is symmetric and vanishes whenever  $z_i = z_j$  for some  $i \neq j$ , hence the desired formula (3.5.5) follows by symmetrizing (3.5.7).  $\square$

In (3.5.4) respectively (3.5.5) the weight function  $\tilde{\Delta}^L(z; q^k)$  respectively  $\tilde{\Delta}^B(z; q^k)$  may be replaced by

$$\hat{\Delta}^L(z; q^k) := \frac{n!}{\Gamma_{q^k}(n+1)} q^{-2k^2 \binom{n}{3} - k(\alpha+1) \binom{n}{2}} \prod_{l=1}^n v_L(z_l) \prod_{1 \leq i < j \leq n} z_i^{2k} (q^{1-k} z_j / z_i; q)_{2k}$$

where  $a = q^\alpha$ , respectively by

$$\hat{\Delta}^B(z; q^k) := \frac{n!}{\Gamma_{q^k}(n+1)} \prod_{l=1}^n v_B(z_l) \prod_{1 \leq i < j \leq n} z_i^{2k} (q^{1-k} z_j / z_i; q)_{2k}.$$

This follows from the fact that  $\tilde{\Delta}^L(\cdot; q^k)$  and  $\tilde{\Delta}^B(\cdot; q^k)$  are symmetric functions satisfying

$$\begin{aligned} \hat{\Delta}^L(z; q^k) &= \frac{n!}{\Gamma_{q^k}(n+1)} \tilde{\Delta}^L(z; q^k) \prod_{i < j} \frac{z_i - q^k z_j}{z_i - z_j}, \\ \hat{\Delta}^B(z; q^k) &= \frac{n!}{\Gamma_{q^k}(n+1)} \tilde{\Delta}^B(z; q^k) \prod_{i < j} \frac{z_i - q^k z_j}{z_i - z_j}, \end{aligned}$$

and from the fact that

$$\sum_{w \in \mathfrak{S}} w \left( \prod_{i < j} \frac{z_i - q^k z_j}{z_i - z_j} \right) = \Gamma_{q^k}(n+1)$$

(cf. [33] or [85, Chapter III, (1.4)]). The  $q$ -Selberg integral (3.3.25) for  $t = q^k$  reduces now to the following evaluation formula.

**COROLLARY 3.5.2.** *Let  $t = q^k$  with  $k \in \mathbb{N}$  and  $(a, b) \in V_L$ . Then*

$$\begin{aligned} &\int_{z_1=0}^1 \cdots \int_{z_n=0}^1 \prod_{1 \leq i < j \leq n} z_i^{2k} (q^{1-k} z_j / z_i; q)_{2k} \prod_{l=1}^n \frac{(qz_l; q)_\infty}{(qbz_l; q)_\infty} z_l^\alpha d_q z_l = \\ &q^{k(\alpha+1) \binom{n}{2} + 2k^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha+1+(j-1)k) \Gamma_q(\beta+1+(j-1)k) \Gamma_q(jk+1)}{\Gamma_q(\alpha+\beta+2+(n+j-2)k) \Gamma_q(k+1)} \end{aligned}$$

where  $a = q^\alpha$ ,  $b = q^\beta$ .

**PROOF.** This follows from (3.3.25), (3.5.4) and the discussion preceding the corollary since

$$\Gamma_{q^k}(n+1) = \prod_{j=1}^n \frac{\Gamma_q(jk+1) \Gamma_q(k)}{\Gamma_q(jk) \Gamma_q(k+1)}.$$

□

Similarly, the  $q$ -Selberg integral (3.4.19) for  $t = q^k$  can be simplified as follows.

COROLLARY 3.5.3. Let  $t = q^k$  with  $k \in \mathbb{N}$  and  $(a, b, c, d) \in V_B$ . Then

$$\begin{aligned} & \int_{z_1=-d}^c \cdots \int_{z_n=-d}^c \prod_{1 \leq i < j \leq n} z_i^{2k} \left( q^{1-k} \frac{z_j}{z_i}; q \right)_{2k} \prod_{l=1}^n \frac{(qz_l/c, -qz_l/d; q)_\infty}{(qaz_l/c, -qbz_l/d; q)_\infty} d_q z_l \\ &= q^{k^2 \binom{n}{3} - \binom{k}{2} \binom{n}{2}} \prod_{j=1}^n \left( \frac{\Gamma_q(\alpha + 1 + (j-1)k) \Gamma_q(\beta + 1 + (j-1)k) \Gamma_q(jk + 1)}{\Gamma_q(\alpha + \beta + 2 + (n+j-2)k) \Gamma_q(k + 1)} \right. \\ & \quad \left. \frac{(-d/c, -c/d; q)_\infty (cd)^{1+(j-1)k}}{(-q^{\alpha+1+(j-1)k}d/c, -q^{\beta+1+(j-1)k}c/d; q)_\infty (c+d)} \right) \end{aligned}$$

where  $a = q^\alpha$  and  $b = q^\beta$ .

PROOF. The proof is similar to the proof of Corollary 3.5.2, using the explicit expression (3.5.1) for the constant  $c_B$  when  $t = q^k$ .  $\square$

Corollaries 3.5.2 and 3.5.3 were conjectured by Askey in [5]. A proof of Corollary 3.5.2 was given independently by Habsieger [33] and Kadell [54]. A proof of Corollary 3.5.3 was given by Evans in [29].

### 3.6. Limit transitions: an algebraic approach

In Section 3.3 and Section 3.4 limit transitions between the Koornwinder polynomials and the big respectively little  $q$ -Jacobi polynomials were derived by proving the limits on the level of the orthogonality measures (see Theorem 3.3.3, Proposition 3.3.2 respectively Theorem 3.4.5, Proposition 3.4.4).

In this section a different, more algebraic method is discussed for deriving limit transitions between families of multivariable orthogonal polynomials. The method is based on the fact that the multivariable orthogonal polynomials under consideration can be characterized as special type of eigenfunctions for certain explicit second order  $q$ -difference (respectively differential) operators. The limit transitions between the families can then be proved by formally computing limits of the operators and their eigenvalues. In particular, the explicit orthogonality relations are not needed for the proof of the limit transitions.

This method will be applied for computing the classical limit  $q \uparrow 1$  of the big and the little  $q$ -Jacobi polynomials. We give also a limit transition from Koornwinder polynomials to generalized Jacobi polynomials which is different from the computation of the classical limit of the Koornwinder polynomials. The method for computing limit transitions between multivariable orthogonal polynomials as presented in this section can also be applied to various other families of multivariable orthogonal polynomials. See for instance [121], where the method is used to derive alternative proofs of the limit transitions from Koornwinder polynomials to multivariable big and little  $q$ -Jacobi polynomials and [19], where the method is used to derive limit transitions from Koornwinder polynomials to multivariable Wilson, to multivariable continuous Hahn polynomials and to  $BC$  type Heckman-Opdam polynomials.

We first recall the definitions of generalized Jacobi polynomials [132] and relate them to the  $BC$  type Heckman Opdam polynomials [36], [37], [38]. Let  $V_J$  be set of parameters  $(\alpha, \beta)$  which satisfy  $\alpha, \beta > -1$ , and let  $\tau > 0$ . Define a positive definite bilinear form  $\langle \cdot, \cdot \rangle_{J, \tau}^{\alpha, \beta}$  on  $A_{\mathbb{R}}^{\mathfrak{S}}$  by

$$\langle f, g \rangle_{J, \alpha, \beta, \tau} := \frac{1}{n!} \int_{z_1=0}^1 \cdots \int_{z_n=0}^1 f(z)g(z)\Delta^J(z; \alpha, \beta; \tau)dz \quad f, g \in A_{\mathbb{R}}^{\mathfrak{S}}$$

with  $\Delta_J(z; \alpha, \beta; \tau) := (\prod_{i=1}^n (1 - z_i)^{\beta} z_i^{\alpha}) |V(z)|^{2\tau}$  and  $V(z) := \prod_{i < j} (z_i - z_j)$  the Vandermonde determinant.

DEFINITION 3.6.1. *Let  $(\alpha, \beta) \in V_J$  and  $\tau > 0$ . The generalized Jacobi polynomials  $P_{\lambda}^J(\cdot; \alpha, \beta; \tau)$  ( $\lambda \in \Lambda$ ) are by definition the unique  $\mathfrak{S}$ -invariant polynomials which satisfy*

- (1)  $P_{\lambda}^J = \tilde{m}_{\lambda} + \sum_{\mu < \lambda; \mu \in \Lambda} d_{\lambda, \mu}^J \tilde{m}_{\mu}$  for some  $d_{\lambda, \mu}^J = d_{\lambda, \mu}^J(\alpha, \beta; \tau) \in \mathbb{R}$   
(2)  $\langle P_{\lambda}^J(\alpha, \beta; \tau), \tilde{m}_{\mu} \rangle_{J, \alpha, \beta, \tau} = 0$  if  $\mu < \lambda$ .

The one variable Jacobi polynomials  $\{P_m^J(z; \alpha, \beta)\}_{m \in \mathbb{Z}_+}$  are independent of  $\tau$  and are explicitly given by

$$(3.6.1) \quad P_m^J(z; \alpha, \beta) = \frac{(-1)^m (\alpha + 1)_m}{(m + \alpha + \beta + 1)_m} {}_2F_1 \left( \begin{matrix} -m, m + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} ; z \right)$$

with

$${}_2F_1 \left( \begin{matrix} a, b \\ c \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

Gauss's hypergeometric series. Usually the Jacobi polynomials are written as functions of  $1 - 2z$  and normalized differently (cf. [28, Section 10.8]).

The generalized Jacobi polynomials are directly related to the Heckman-Opdam polynomials of type  $BC$  (cf. [36], [37], [38]) as follows. Using the notations introduced in Remark 2.3.5, we set  $\langle \cdot, \cdot \rangle$  for the standard inner product on  $\mathbb{C}^n$ , so  $\langle e_i, e_j \rangle = \delta_{i,j}$ . A multiplicity function  $k$  is a function  $k : R \rightarrow \mathbb{C}$  such that  $k_{\alpha} = k_{w\alpha}$  for all  $\alpha \in R, w \in \mathcal{W}$ .  $k = (k_{\alpha})_{\alpha \in R}$  is completely determined by  $k_1 := k_{e_1}, k_2 := k_{e_1 + e_2}$  and  $k_3 := k_{2e_1}$ , so we will sometimes write  $k = (k_1, k_2, k_3)$ . Let  $V_{HO}$  be the set of parameters  $(k_1, k_2, k_3)$  such that  $k_1 + k_3 > -\frac{1}{2}, k_3 > -\frac{1}{2}$  and  $k_2 > 0$ . Define a positive definite bilinear form on  $A_{\mathbb{R}}^{\mathcal{W}} := \mathbb{R}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]^{\mathcal{W}}$  for  $k \in V_{HO}$  by

$$\langle f, g \rangle_k := \int_{\theta_1=0}^{2\pi} \cdots \int_{\theta_n=0}^{2\pi} f(e^{i\theta})g(e^{i\theta})\Delta^{HO}(\theta; k)d\theta,$$

with  $e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_n})$  and weight function

$$\begin{aligned} \Delta^{HO}(\theta; k) &:= \prod_{\alpha \in R} \left( e^{\frac{1}{2}i\langle \alpha, \theta \rangle} - e^{-\frac{1}{2}i\langle \alpha, \theta \rangle} \right)^{k_{\alpha}} \\ &= c(k) \prod_{j=1}^n (\sin^2(\theta_j/2))^{k_1 + k_3} (\cos^2(\theta_j/2))^{k_3} \prod_{l < m} |\sin^2(\theta_l/2) - \sin^2(\theta_m/2)|^{2k_2} \end{aligned}$$

where  $c(k) = 4^{n(k_1+2k_3)+n(n-1)k_2}$ .

**DEFINITION 3.6.2.** Let  $k = (k_1, k_2, k_3) \in V_{HO}$ . The BC type Heckman-Opdam polynomials  $\{P_\lambda^{HO}(z; k)\}_{\lambda \in \Lambda}$  are by definition the unique  $\mathcal{W}$ -invariant Laurent polynomials which satisfy

- (1)  $P_\lambda^{HO} = m_\lambda + \sum_{\mu < \lambda; \mu \in \Lambda} d_{\lambda, \mu}^{HO} m_\mu$  for some  $d_{\lambda, \mu}^{HO} \in \mathbb{R}$ ,
- (2)  $\langle P_\lambda^{HO}(\cdot; k), m_\mu \rangle_k = 0$  if  $\mu < \lambda$ .

Observe that

$$\tilde{m}_\lambda(\sin^2(\theta/2)) = (-4)^{-|\lambda|} m_\lambda(e^{i\theta}) + \sum_{\mu < \lambda} b_{\lambda, \mu} m_\mu(e^{i\theta})$$

for certain constants  $b_{\lambda, \mu}$ , where  $\sin^2(\theta/2) := (\sin^2(\theta_1/2), \dots, \sin^2(\theta_n/2))$ . It follows that the defining conditions for  $P_\lambda^J$  (Definition 3.6.1) with  $\alpha = k_1 + k_3 - \frac{1}{2}, \beta = k_3 - \frac{1}{2}$  and  $\tau = k_2$  become the defining conditions for  $P_\lambda^{HO}(k)$  (Definition 3.6.2) under the change of variables  $x_i := \sin^2(\theta_i/2)$  ( $i = 1, \dots, n$ ), up to the constant  $(-4)^{|\lambda|}$ . So the relation between Heckman-Opdam polynomials of type BC and the generalized Jacobi polynomials is given by

$$(3.6.2) \quad P_\lambda^{HO}(e^{i\theta}; k) = (-4)^{|\lambda|} P_\lambda^J\left(\sin^2(\theta/2); k_1 + k_3 - \frac{1}{2}, k_3 - \frac{1}{2}, k_2\right)$$

for  $\lambda \in \Lambda$ . Set  $\partial_j := \frac{\partial}{\partial z_j}$  and let  $D_{J, \tau}^{\alpha, \beta}$  be the second order partial differential operator given by

$$(3.6.3) \quad D_{J, \alpha, \beta, \tau} := \sum_{j=1}^n ((z_j - 1)z_j \partial_j^2 + ((2 + \alpha + \beta)z_j - (\alpha + 1))\partial_j) + 2\tau(z_j - 1)z_j V(z)^{-1} (\partial_j V)(z) \partial_j.$$

**PROPOSITION 3.6.3.** ([132]) Fix  $\lambda \in \Lambda$ . For arbitrary  $\alpha, \beta, \tau \in \mathbb{C}$  there exist constants  $E_{\lambda, \mu}^J(\alpha, \beta; \tau) \in \mathbb{C}$  ( $\mu \leq \lambda$ ) depending polynomially on  $\alpha, \beta$  and  $\tau$ , such that

$$D_J m_\lambda = \sum_{\mu \leq \lambda} E_{\lambda, \mu}^J m_\mu.$$

The leading term  $E_{\lambda, \lambda}^J(\alpha, \beta; \tau)$  will be denoted by  $E_\lambda^J(\alpha, \beta; \tau)$  and is explicitly given by

$$(3.6.4) \quad E_\lambda^J(\alpha, \beta; \tau) := \sum_{j=1}^n \lambda_j (\lambda_j + \alpha + \beta + 1 + 2(n - j)\tau).$$

**PROOF.** This can be proved by a straightforward calculation (compare with [132, p. 817]).  $\square$

Furthermore,  $D_{J, \alpha, \beta, \tau}$  is symmetric with respect to  $\langle \cdot, \cdot \rangle_{J, \alpha, \beta, \tau}$  for  $(\alpha, \beta) \in V_J$  and  $\tau > 0$  (see [132, Theorem 4.3]). It follows from the definition of  $P_\lambda^J$ , Proposition 3.6.3 and the symmetry of  $D_J$  that

$$(3.6.5) \quad D_{J, \alpha, \beta, \tau} P_\lambda^J(\cdot; \alpha, \beta; \tau) = E_\lambda^J(\alpha, \beta; \tau) P_\lambda^J(\cdot; \alpha, \beta; \tau)$$

for all  $\lambda \in \Lambda$  when  $(\alpha, \beta) \in V_J$  and  $\tau > 0$ . Furthermore, the generalized Jacobi polynomials  $\{P_\lambda^J\}_{\lambda \in \Lambda}$  are mutual orthogonal with respect to  $\langle \cdot, \cdot \rangle_J$  (see [36], [37] and [38], where full orthogonality is established for Heckman-Opdam polynomials associated with arbitrary root systems).

In Section 2.3 it was observed that the Koornwinder polynomials can be expressed in terms of the second order  $q$ -difference operator  $D_{\underline{t}, t}$  (2.3.13) and its eigenvalues by the formula

$$(3.6.6) \quad P_\lambda(\cdot; \underline{t}; t) = \left( \prod_{\mu < \lambda} \frac{D_{\underline{t}, t} - E_\mu(\underline{t}; t)}{E_\lambda(\underline{t}; t) - E_\mu(\underline{t}; t)} \right) m_\lambda$$

for parameters  $\underline{t}$  and  $t$  such that  $E_\mu(\underline{t}; t) \neq E_\lambda(\underline{t}; t)$  for all  $\mu < \lambda$  ( $E_\lambda$  given by (2.3.16)). Formula (3.6.6) can be used to extend the definition of the Koornwinder polynomial of degree  $\lambda$  to arbitrary parameter values  $(\underline{t}, t) \in \mathbb{C}^5$  for which  $E_\mu(\underline{t}; t) \neq E_\lambda(\underline{t}; t)$  for  $\mu < \lambda$ . The Koornwinder polynomials  $P_\lambda(\cdot; \underline{t}; t)$  ( $\lambda \in \Lambda$ ) for generic parameters  $(\underline{t}, t) \in \mathbb{C}^5$  then do no longer have interpretations as orthogonal polynomials, but they are still characterized as the unique  $\mathcal{W}$ -invariant Laurent polynomials of the form  $P_\lambda(\cdot; \underline{t}; t) = m_\lambda + \sum_{\mu < \lambda} c_{\lambda, \mu}(\underline{t}; t) m_\mu$  which are eigenfunctions of  $D_{\underline{t}, t}$  with eigenvalues  $E_\lambda(\underline{t}; t)$  ( $\lambda \in \Lambda$ ).

Similarly, the multivariable big and little  $q$ -Jacobi polynomials as well as the generalized Jacobi polynomials of degree  $\lambda$  can be expressed as

$$(3.6.7) \quad P_\lambda^X = \left( \prod_{\mu < \lambda} \frac{D_X - E_\mu^X}{E_\lambda^X - E_\mu^X} \right) \tilde{m}_\lambda, \quad X = B, L \text{ resp. } J$$

provided that the eigenvalues satisfy  $E_\mu^X \neq E_\lambda^X$  for all  $\mu < \lambda$ , where  $E_\lambda^B = E_\lambda^L := E_\lambda^{qJ}$  (3.3.23). Formula (3.6.7) for the generalized Jacobi polynomials follows from the triangularity of  $D_J$  (Proposition 3.6.3). Formula (3.6.7) for the little and big  $q$ -Jacobi polynomials follows from the fact that  $D_L$  respectively  $D_B$  is triangular with respect to the monomials  $\{\tilde{m}_\lambda\}_{\lambda \in \Lambda}$  and with respect to the dominance order  $\leq$ , with leading terms given by the eigenvalues  $\{E_\lambda^{qJ}\}_{\lambda \in \Lambda}$  (this in turn follows from Corollary 3.3.5 and Corollary 3.4.7). This triangularity property holds in fact for arbitrary complex parameters  $a, b, c, d, t$  by analytic continuation. Formula (3.6.7) will be used as definition of the polynomial  $P_\lambda^X$  for all parameter values such that  $E_\lambda^X \neq E_\mu^X$  if  $\mu < \lambda$ .

**REMARK 3.6.4.** The triangularity of  $D_B$  and  $D_L$  can also be proved by direct computations for arbitrary complex parameter values  $a, b, c, d$  and  $t$  without using any information about their eigenfunctions (one only needs some elementary properties of the Schur functions). For details, see [116, Proposition 4.2].

There is another way of looking at the above mentioned extended definitions of multivariable polynomials. For instance, consider the Koornwinder polynomial  $P_\lambda =$

$m_\lambda + \sum_{\mu < \lambda} d_{\lambda, \mu} m_\mu$  of degree  $\lambda$ , then Proposition 2.3.6 and (3.6.6) imply that the coefficients  $d_{\lambda, \mu} = d_{\lambda, \mu}(\underline{t}, t)$  depend rationally on the parameters  $(\underline{t}, t)$ . The extended definition of the Koornwinder polynomials to generic parameter values  $(\underline{t}, t) \in \mathbb{C}^5$  corresponds with extending  $d_{\lambda, \mu}$  as rational function of  $(\underline{t}, t)$  for all  $\mu < \lambda$ .

Using the formulas (3.6.6) and (3.6.7), limit transitions between multivariable orthogonal polynomials can now be proved by computing limits of the corresponding  $q$ -difference operators. We first give sufficient conditions on the parameter sets such that the eigenvalues of the triangular operators are mutually different for compatible weights (compatible with respect to the dominance order  $\leq$ ). The first part of the proposition is a slight extension of [16, Lemma 5.1].

**PROPOSITION 3.6.5.** *Let  $\lambda, \mu \in \Lambda$  with  $\mu < \lambda$ .*

- (i)  $E_\mu(\underline{t}; t) < E_\lambda(\underline{t}; t)$  if  $t_0 t_1 t_2 t_3 \in [-q, 1)$  and  $t \in (0, 1)$ ;
- (ii)  $E_\mu^{qJ}(a, b; t) < E_\lambda^{qJ}(a, b; t)$  if  $ab \in [-q^{-1}, q^{-2})$  and  $t \in (0, 1)$ ;
- (iii)  $E_\mu^J(\alpha, \beta; \tau) < E_\lambda^J(\alpha, \beta; \tau)$  for  $(\alpha, \beta) \in V_J$  and  $\tau > 0$ .

For the proof of the proposition, we use the following fact.

**LEMMA 3.6.6.** *Let  $\mu, \lambda \in \Lambda$  with  $\mu \leq \lambda$ . Then we can walk from  $\mu$  to  $\lambda$  while staying within  $\Lambda$  by successively adding an element of the subset  $\tilde{R}^+ := \{e_i\}_{i=1}^n \cup \{e_i - e_j\}_{i < j}$  of the positive roots.*

**PROOF.** Let  $\mu, \lambda \in \Lambda$  such that  $\mu < \lambda$ . It suffices to prove that there exists an  $\alpha \in \tilde{R}^+$  such that  $\mu + \alpha \in \Lambda$  and  $\mu + \alpha \leq \lambda$ . Let

$$\lambda - \mu = \sum_{i=1}^{n-1} c_i (e_i - e_{i+1}) + c_n e_n$$

with  $c_j \in \mathbb{Z}_+$  be the decomposition of  $\lambda - \mu$  with respect to the simple roots  $S = \{e_i - e_{i+1}\}_{i=1}^{n-1} \cup \{e_n\}$ . So we have that  $\lambda_k - \mu_k = c_k - c_{k-1}$  for  $k = 2, \dots, n$  and  $\lambda_1 - \mu_1 = c_1$ . Furthermore we have

$$\sum_{j=1}^i (\lambda_j - \mu_j) = c_i \quad i \in [1, n].$$

Let  $\{c_p, \dots, c_{q-1}\}$  ( $p < q$ ) be a string such that  $c_j > 0$  for  $j = p, \dots, q-1$  and such that  $c_{p-1} = 0$  (or  $p = 1$ ) and  $c_q = 0$  (or  $q = n+1$ ). Then  $\mu_{p-1} \geq \lambda_{p-1} \geq \lambda_p > \mu_p$  (or  $p = 1$  and  $\lambda_p > \mu_p$ ) and  $\mu_q > \lambda_q \geq \lambda_{q+1} \geq \mu_{q+1}$  (or  $q = n+1$ , or  $q = n$  and  $\mu_q > \lambda_q$ ). So  $\alpha = e_p - e_q \in \tilde{R}^+$  does the job for  $q < n+1$ , and  $\alpha = e_p \in \tilde{R}^+$  for  $q = n+1$ .  $\square$

**PROOF OF PROPOSITION 3.6.5.** The inequality (ii) follows immediately from (i). For the proof of (i) and (iii) it suffices to check the inequalities for the special case that  $\lambda - \mu \in \tilde{R}^+$  in view of Lemma 3.6.6. The proof follows then by direct computation, using the explicit expressions of the eigenvalues  $E_\lambda$  (2.3.16) and  $E_\lambda^J$  (3.6.4). As an example, we show that  $E_\mu < E_\lambda$  when  $\lambda - \mu = e_i - e_j$  and  $1 \leq i < j \leq n$ . Then  $\lambda_i = \mu_i + 1$ ,

$\lambda_j = \mu_j - 1$  and  $\lambda_k = \mu_k$  for  $k \neq i, j$ . It follows that the difference  $E_\lambda(\underline{t}; t) - E_\mu(\underline{t}; t)$  is equal to

$$(1 - q)t^{i-1}q^{-\mu_i-1} (1 - t^{j-i}q^{\mu_i-\mu_j+1}) (1 + t_0t_1t_2t_3t^{2n-i-j}q^{\mu_i+\mu_j-1}).$$

Since  $i < j$  and  $\mu_i \geq \mu_j$ , we have  $E_\lambda(\underline{t}; t) - E_\mu(\underline{t}; t) > 0$  for all  $t \in (0, 1)$  if  $t_0t_1t_2t_3 \geq -q$ .  $\square$

For  $\gamma, \delta \in \mathbb{C}$ , we write  $\gamma z + \delta = (\gamma z_1 + \delta, \dots, \gamma z_n + \delta)$  and  $z^{-1} = (z_1^{-1}, \dots, z_n^{-1})$ .

**THEOREM 3.6.7.** *Let  $(\alpha, \beta) \in V_J$ ,  $\tau > 0$  and  $\lambda \in \Lambda$ .*

**(i)** *Let  $c, d \in \mathbb{C}$  such that  $c, d \neq 0$ ,  $c^2 \neq d$ ,  $d \neq 1$ . Then*

(3.6.8)

$$\lim_{q \uparrow 1} P_\lambda \left( z; c, \frac{q^{\alpha+1}}{c}, \frac{q^{\beta+1}d}{c}, \frac{c}{d}; q^\tau \right) = k_{|\lambda|}^{c,d} P_\lambda^J \left( \frac{1+c^2-c(z+z^{-1})}{(1-d)(1-c^2/d)}; \alpha, \beta; \tau \right)$$

with  $k_m^{c,d} := \left( \frac{(d-1)(1-c^2/d)}{c} \right)^m$ .

**(ii)** *Let  $c, d > 0$ , then*

$$(3.6.9) \quad \lim_{q \uparrow 1} P_\lambda^B(z; q^\alpha, q^\beta, c, d; q^\tau) = (-1)^{|\lambda|} (c+d)^{|\lambda|} P_\lambda^J \left( \frac{c-z}{c+d}; \alpha, \beta; \tau \right).$$

**(iii)** *We have*

$$(3.6.10) \quad \lim_{q \uparrow 1} P_\lambda^L(z; q^\alpha, q^\beta; q^\tau) = P_\lambda^J(z; \alpha, \beta; \tau).$$

These limit transitions should be interpreted as pointwise limits as described in Section 3.3 (after Proposition 3.3.2). The multivariable polynomials involved in the three limit transitions are well defined. Indeed, for the limits from big and little  $q$ -Jacobi polynomials to generalized Jacobi polynomials, the corresponding parameters satisfy  $(q^\alpha, q^\beta, c, d) \in V_B$  (cf. Definition 3.2.3) respectively  $(q^\alpha, q^\beta) \in V_L$  (cf. Definition 3.2.1). In fact, one has  $q^{\alpha+\beta} \in [0, q^{-2})$  in both cases, so the multivariable big  $q$ -Jacobi polynomials involved in the limit (3.6.9) are well defined for all  $c, d \in \mathbb{C}$  with  $c+d \neq 0$  in view of Proposition 3.6.5(ii). So the conditions on  $c, d$  for the limit transition (3.6.9) may be weakened to the condition  $c+d \neq 0$ .

For the limit from Koornwinder polynomials to generalized Jacobi polynomials, the product of the four parameters  $\underline{t}$  equals  $q^{\alpha+\beta+2} \in [0, 1)$ , hence the Koornwinder polynomials involved in the limit (3.6.8) are well defined by Proposition 3.6.5(i).

**REMARK 3.6.8.** In the one variable case ( $n = 1$ ), the three limit transitions given in Theorem 3.6.7 can easily be derived from the explicit expressions of the polynomials in terms of (basic) hypergeometric series.

The limit transition (3.6.8) from Koornwinder polynomials to generalized Jacobi polynomials is different from taking the classical limit of the Koornwinder polynomials.



Indeed, the classical limit of the Koornwinder polynomials is given by

$$(3.6.11) \quad \lim_{q \uparrow 1} P_\lambda(z; q^{g_0}, -q^{g_1}, q^{g'_0+1/2}, -q^{g'_1+1/2}; q^g) \\ = P_\lambda^{HO}(z; g_0 + g'_0 - g_1 - g'_1, g, g_1 + g'_1)$$

(see, for instance, [19]), where we assume that  $g > 0$  and  $g_0 + g'_0, g_1 + g'_1 > -1/2$ .

To illustrate the computations involved in proving Theorem 3.6.7, we will give a detailed proof of the limit transition (3.6.9) from big  $q$ -Jacobi polynomials to generalized Jacobi polynomials. Under the change of variables  $z_i = \frac{c-y_i}{c+d}$  ( $i \in [1, n]$  and  $c + d \neq 0$ ), the second order differential operator  $D_{J,\tau}^{\alpha,\beta}$  becomes

$$(3.6.12) \quad D_{BJ,\alpha,\beta,c,d,\tau} := \sum_{j=1}^n \left( (y_j - c)(y_j + d)\partial_j^2 + \hat{B}_j(y)\partial_j \right)$$

where  $\hat{B}_j$  is given by

$$(3.6.13) \quad \hat{B}_j(y) := 2\tau(y_j - c)(y_j + d)V(y)^{-1} (\partial_j V)(y) \\ + (2 + \alpha + \beta)y_j + d(\alpha + 1) - c(\beta + 1).$$

Denote the right hand side of (3.6.9) by  $P_\lambda^{BJ}(z) := P_\lambda^{BJ}(z; \alpha, \beta; c, d; \tau)$ , so

$$(3.6.14) \quad P_\lambda^{BJ}(z) := (-1)^{|\lambda|} (c+d)^{|\lambda|} P_\lambda^J \left( \frac{c-z}{c+d}; \alpha, \beta; \tau \right)$$

for  $\lambda \in \Lambda$ .

LEMMA 3.6.9. For  $(\alpha, \beta) \in V_J$ ,  $c + d \neq 0$  and  $\tau > 0$  we have

$$P_\lambda^{BJ} = \left( \prod_{\mu < \lambda} \frac{D_{BJ} - E_\mu^J}{E_\lambda^J - E_\mu^J} \right) \tilde{m}_\lambda$$

for all  $\lambda \in \Lambda$ , where  $E_\lambda^J$  (independent of  $c$  and  $d$ ) is given by (3.6.4).

PROOF. For  $c_1, c_2 \in \mathbb{C}$  we have

$$(3.6.15) \quad \tilde{m}_\lambda(c_1 + c_2 y_1, \dots, c_1 + c_2 y_n) = c_2^{|\lambda|} \tilde{m}_\lambda(y) + \sum_{\mu < \lambda} b_{\lambda,\mu} \tilde{m}_\mu(y)$$

for certain  $b_{\lambda,\mu} \in \mathbb{C}$  depending on  $c_1$  and  $c_2$ . Hence  $P_\lambda^{BJ}$  (3.6.14) satisfies

$$(3.6.16) \quad P_\lambda^{BJ} = \tilde{m}_\lambda + \sum_{\mu < \lambda} d_{\lambda,\mu}^{BJ} \tilde{m}_\mu$$

for certain  $d_{\lambda,\mu}^{BJ} \in \mathbb{C}$ . By (3.6.5) we have the eigenvalue equation

$$(3.6.17) \quad D_{BJ} P_\lambda^{BJ} = E_\lambda^J P_\lambda^{BJ}$$

for all  $\lambda \in \Lambda$ . Hence  $D_{BJ}$  is triangular with respect to the monomials  $\{\tilde{m}_\lambda\}_{\lambda \in \Lambda}$  and the dominance order  $\leq$ , with leading terms given by the eigenvalues  $\{E_\lambda^J\}_{\lambda \in \Lambda}$ . The proof

follows since the eigenvalues  $E_\lambda^J$  are mutual different for compatible weights (Proposition 3.6.5(iii)).  $\square$

The proof of Theorem 3.6.7(ii) is now a direct consequence of the previous lemma, (3.6.7) for  $X = B$  and the following lemma.

LEMMA 3.6.10. *Let  $(\alpha, \beta) \in V_J$  and  $\tau > 0$ . For  $c + d \neq 0$  we have*

$$(3.6.18) \quad \lim_{q \uparrow 1} \frac{(D_{B,q^\alpha,q^\beta,c,d,q^\tau} f)(z)}{(1-q)^2} = (D_{BJ,\alpha,\beta,c,d,\tau} f)(z) \quad \forall f \in A^\mathfrak{S}.$$

The eigenvalues satisfy

$$(3.6.19) \quad \lim_{q \uparrow 1} \frac{E_\lambda^B(q^\alpha, q^\beta, c, d; q, q^\tau)}{(1-q)^2} = E_\lambda^J(\alpha, \beta, c, d; \tau) \quad \forall \lambda \in \Lambda.$$

PROOF. The limit (3.6.18) makes sense (in terms of the definition of limit transitions as given in Section 3.3 after Proposition 3.3.2) since  $D_B$  (as well as  $D_{BJ}$ ) preserves  $A^\mathfrak{S}$ . So it suffices to prove the limit transition (3.6.18) pointwise for generic  $z \in \mathbb{C}^n$ . We therefore assume throughout the proof that  $z \in (\mathbb{C}^*)^n$  and  $V(z) \neq 0$ .

The second order  $q$ -difference operator  $D_B$  (3.4.17) can then be rewritten as

$$(3.6.20) \quad (D_B f)(z) = \sum_{i=1}^n \left( A_i(z) (T_i^- ((D_q^{i,-})^2 f))(z) + B_i(z) (T_i^- (D_q^{i,-} f))(z) \right),$$

with  $D_q^{i,-}$  the backward partial  $q$ -derivative in direction  $i$ ,

$$(D_q^{i,-} f)(z) := \frac{(f - T_i^+ f)(z)}{(1-q)z_i}$$

and  $A_i(z) = A_i(z; a, b, c, d; t)$  respectively  $B_i(z) = B_i(z; a, b, c, d; t)$  given by

$$\begin{aligned} A_i(z) &= q^{-1}(1-q)^2 z_i^2 \phi_{B,i}^+(z) \\ &= (1-q)^2 q^{-2} (qaz_i - c)(qbz_i + d) t^{n-1} V(z)^{-1} (T_{t,i} V)(z), \\ B_i(z) &= q^{-1}(1-q)z_i (\phi_{B,i}^-(z) - q\phi_{B,i}^+(z)) \\ &= \frac{(1-q)}{q} t^{n-1} ((z_i + (d-c) - cdz_i^{-1})V(z)^{-1} (T_{t-1,i} V)(z) \\ &\quad - (q^2 abz_i + (qad - qbc) - cdz_i^{-1})V(z)^{-1} (T_{t,i} V)(z)) \end{aligned}$$

where  $\phi_{B,i}^\pm$  are given by (3.4.18) and with  $T_{t^\pm 1,i}$  the multiplicative  $t^{\pm 1}$ -shift in the variable  $z_i$ . It is immediate that

$$(3.6.21) \quad \lim_{q \uparrow 1} \frac{A_i(z; q^\alpha, q^\beta, c, d; q^\tau)}{(1-q)^2} = (z_i - c)(z_i + d).$$

To evaluate the limit for  $B_i$ , we consider for a complex variable  $x$  the map

$$(D_{u,v}^q f)(x) := \frac{(f(q^{-u}x) - f(q^v x))}{(1-q)x}, \quad (u, v \in \mathbb{R}).$$

Observe that  $D_{u,v}^q$  maps polynomials to polynomials respectively Laurent polynomials to Laurent polynomials, and that

$$(3.6.22) \quad \lim_{q \uparrow 1} (D_{u,v}^q f)(x) = (u+v) \frac{df}{dx}(x) \quad \forall f \in \mathbb{C}[x, x^{-1}].$$

In particular,

$$\lim_{q \uparrow 1} \frac{(T_{q^{-\tau}, i} V - T_{q^\tau, i} V)(z)}{(1-q)z_j} = 2\tau(\partial_i V)(z).$$

It follows now by a straightforward computation that

$$(3.6.23) \quad \lim_{q \uparrow 1} \frac{B_i(z; q^\alpha, q^\beta, c, d; q^\tau)}{(1-q)^2} = \hat{B}_i(z)$$

where  $\hat{B}_i$  is given by (3.6.13). The limit (3.6.18) follows now from (3.6.12), (3.6.20), (3.6.21), (3.6.22) and (3.6.23) since  $D_{0,1}^q$  is the backward partial  $q$ -derivative in the variable  $x$ . The limit (3.6.19) is checked by direct computation. It can also be proved without computation using (3.6.18) and using the triangularity properties of  $D_{B,J}$  (Lemma 3.6.9) and of  $D_B$ .  $\square$

The proof of the limit transition (3.6.8) from Koornwinder polynomials to generalized Jacobi polynomials is similar, but the computations are more involved due to the less trivial change of variables  $z_i = \frac{1+c^2-c(y_i+y_i^{-1})}{(1-d)(1-c^2/d)}$  ( $i \in [1, n]$ ) which is needed. The only essential difference in the proof is then the proper analogue of (3.6.15), which is given by

$$\tilde{m}_\lambda \left( \frac{1+c^2-c(y+y^{-1})}{(1-d)(1-c^2/d)} \right) = (k_{|\lambda|}^{c,d})^{-1} m_\lambda(y) + \sum_{\mu < \lambda} b_{\lambda,\mu} m_\mu(y)$$

for certain  $b_{\lambda,\mu} \in \mathbb{C}$ , where  $k_m^{c,d}$  is defined in Theorem 3.6.7(i) (note that this formula rewrites an  $\mathfrak{S}$ -invariant monomial as linear combination of  $\mathcal{W}$ -invariant monomials  $m_\mu$  ( $\mu \leq \lambda$ ) with respect to the new variables  $y$ ). The limit transition (3.6.10) from little  $q$ -Jacobi polynomials to generalized Jacobi polynomials is proved similarly as the limit transition (3.6.9), the computations being simpler since no change of variables is needed. In fact, the analogue of Lemma 3.6.10 in the little  $q$ -Jacobi case is a special case of Lemma 3.6.10, since

$$(3.6.24) \quad D_{L,a,b,t} = D_{B,b,a,1,0,t},$$

where  $D_L$  is given by (3.3.21). In fact, (3.6.24) together with (3.6.7) for  $X = B, L$  immediately implies the limit transition

$$(3.6.25) \quad \lim_{d \downarrow 0} P_\lambda^B(z; b, a, 1, d; t) = P_\lambda^L(z; a, b; t)$$

for  $\lambda \in \Lambda$ ,  $(a, b) \in V_L$  and  $t \in (0, 1)$  such that  $E_\mu^{q,J}(a, b; t) \neq E_\lambda^{q,J}(a, b; t)$  for all  $\mu < \lambda$ . By a rather tricky combination of a rationality argument with respect to the parameter  $d$  and a continuity argument in  $t$ , a proof of the limit transition (3.6.25) can be given for all  $\lambda \in \Lambda$  and all parameter values  $(a, b) \in V_L$  and  $t \in (0, 1)$  (cf. [121, Theorem 6.3]). In the

proof of [121, Theorem 6.3] the orthogonality relations of the big  $q$ -Jacobi polynomials is used in an essential way.

For a more detailed discussion of the limit transitions (3.6.8), (3.6.9) and (3.6.25) we refer the reader to [121].

### 3.7. Limit of the orthogonality measure (little $q$ -Jacobi case)

In this section a proof of Proposition 3.3.2 is given for parameters  $(a, b) \in V_L$  with  $b \neq 0$  and  $a \leq q^{-\frac{1}{2}}$ . The condition  $b \neq 0$  is not essential, we only make this assumption because the formulas are more transparent when we may divide by  $b$ . For  $b = 0$  all formulas can be rewritten in an obvious way such that they are meaningful and correct. The assumption  $a \leq q^{-\frac{1}{2}}$  is less harmless. Under this assumption the parameter  $t_{L,1}(\varepsilon) = -aq^{\frac{1}{2}}$  (3.2.1) has modulus  $\leq 1$  and hence does not contribute to the discrete parts of the symmetric form  $\langle \cdot, \cdot \rangle_{\underline{t}_L(\varepsilon), t}$  (2.6.11) in the limit (3.3.7). When  $a \in (q^{-\frac{1}{2}}, q^{-1})$ , then the set of mass points  $D_1(m; \underline{t}_L(\varepsilon); t)$  (2.6.2) in the discrete parts of  $\langle \cdot, \cdot \rangle_{\underline{t}_L(\varepsilon), t}$  is  $\varepsilon$ -independent since  $t_{L,1}(\varepsilon)$  is  $\varepsilon$ -independent. It can be shown that any (partly discrete) contribution to the mass points of  $\langle \cdot, \cdot \rangle_{\underline{t}_L(\varepsilon), t}$  with some of its coordinates living in  $D_1(m; \underline{t}_L(\varepsilon); t)$  with  $m > 0$ , vanishes in the limit (3.3.7). In other words, we may as well assume that there are no discrete contributions coming from the term  $t_{L,1}(\varepsilon)$ , which corresponds to the case  $a \leq q^{-\frac{1}{2}}$ . For a detailed discussion with  $a > q^{-\frac{1}{2}}$ , we refer the reader to [119].

So under these assumptions, only  $t_{L,0}(\varepsilon) = \varepsilon^{-1}q^{\frac{1}{2}}$  contributes to the discrete parts of the symmetric form  $\langle \cdot, \cdot \rangle_{\underline{t}_L(\varepsilon), t}$  in the limit (3.3.7). Set  $\rho_{L,j}(\varepsilon) := t_{L,0}(\varepsilon)t^{j-1} = \varepsilon^{-1}q^{\frac{1}{2}}t^{j-1}$  for  $j \in \mathbb{Z}$ . Then  $F(r; \underline{t}_L(\varepsilon); t)$  (2.6.3) for  $\varepsilon > 0$  sufficiently small is given by

$$F(r; \underline{t}_L(\varepsilon); t) = D_0(r; \underline{t}_L(\varepsilon); t) \subset \mathbb{C}^r$$

with the set  $D_0(r; \underline{t}_L(\varepsilon); t)$  (2.6.2) for  $r > 0$  given by

$$(3.7.1) \quad \begin{aligned} D_0(r; \underline{t}_L(\varepsilon); t) &= \{\rho_L(\varepsilon)q^\nu \mid \nu \in P_L(r; \varepsilon)\}, \\ P_L(r; \varepsilon) &:= \{\nu \in P(r) \mid |\rho_{L,0}(\varepsilon)q^{\nu_r}| > 1\}. \end{aligned}$$

Using the explicit definition of the symmetric form  $\langle \cdot, \cdot \rangle$  (2.6.11) as given in Section 2.6, as well as the definition for  $m_\lambda(z|u)$  (3.3.8), we can write

$$(3.7.2) \quad \begin{aligned} & \left( \prod_{i=1}^n (-\varepsilon^{-1}qt^{i-1}, -\varepsilon^{-1}qat^{i-1}; q)_\infty \right) (\varepsilon q^{-1/2})^{|\lambda|+|\mu|} \langle m_\lambda, m_\mu \rangle_{\underline{t}_L(\varepsilon), t} \\ &= \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \sum_{r, \nu} \iint_{z \in T^{n-r}} (m_\lambda m_\mu) (\rho_L q^\nu, \varepsilon q^{-\frac{1}{2}} z | \varepsilon q^{-\frac{1}{2}}) \mathcal{W}_r^L(\nu, z; \varepsilon) \frac{dz}{z} \end{aligned}$$

where the sum is over pairs  $(r, \nu)$  with  $r \in [0, n]$  and  $\nu \in P(r)$  (the sum over  $\nu \in P(r)$  should be ignored when  $r = 0$ ) and with  $\rho_{L,j} := t^{j-1}$ . The renormalized weights

$\mathcal{W}_r^L(\nu, z; \varepsilon)$  are defined by  $\mathcal{W}_0^L(-; z; \varepsilon) := \Delta(z; \underline{t}_L(\varepsilon); t)$  for  $r = 0$ , and for  $r \in [1, n]$ ,

$$\mathcal{W}_r^L(\nu, z; \varepsilon) = \left( \prod_{i=1}^n (-\varepsilon^{-1}qt^{i-1}, -\varepsilon^{-1}qat^{i-1}; q)_\infty \right) \Delta_r^K(\rho_L(\varepsilon)q^\nu, z; \underline{t}_L(\varepsilon); t)$$

if  $\nu \in P_L(r; \varepsilon)$  and zero otherwise. Split the renormalized weights in two parts,

$$(3.7.3) \quad \mathcal{W}_r^L(\nu, z; \varepsilon) = \Delta_{1,r}^{KL}(\nu; \varepsilon) \Delta_{2,r}^{KL}(\nu, z; \varepsilon)$$

where  $\Delta_{1,r}^{KL}$  and  $\Delta_{2,r}^{KL}$  are given by

$$\Delta_{1,r}^{KL}(\nu; \varepsilon) := \left( \prod_{i=1}^r (-\varepsilon^{-1}qt^{i-1}, -\varepsilon^{-1}qat^{i-1}; q)_\infty \right) \Delta^{(d)}(\rho_L(\varepsilon)q^\nu; \underline{t}_{L,0}(\varepsilon))$$

if  $\nu \in P_L(r; \varepsilon)$  and zero otherwise, respectively

$$\Delta_{2,r}^{KL}(\nu, z; \varepsilon) := \prod_{i=1}^{n-r} (-\varepsilon^{-1}qt^{r+i-1}, -\varepsilon^{-1}qat^{r+i-1}; q)_\infty \cdot \Delta(z; \underline{t}_L(\varepsilon); t) \delta_c(\rho_L(\varepsilon)q^\nu; z)$$

if  $\nu \in P_L(r; \varepsilon)$ ,  $z \in T^{n-r}$  and zero otherwise, where  $\Delta$  is given by (2.3.4),  $\Delta^{(d)}$  is given by (2.4.7) and  $\delta_c$  is given by (2.4.11). We have used here the obvious convention that  $\Delta_{1,0}^{KL}(-; \varepsilon) = 1$  and  $\Delta_{2,n}^{KL}(\nu, -; \varepsilon) = 1$  if  $\nu \in P_L(r; \varepsilon)$ .

Lebesgue's dominated convergence Theorem will now be used to pull a limit  $\varepsilon_k \downarrow 0$  in the right hand side of (3.7.2) through the integration over  $z \in T^{n-r}$  and through the infinite sum over  $\nu \in P(r)$  for some sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to 0. For the application of the Lebesgue's dominated convergence Theorem we need certain estimates for the functions  $\Delta_{1,r}^{KL}$  and  $\Delta_{2,r}^{KL}$ , which are given in the following lemma.

LEMMA 3.7.1. *Keep the notations and conventions as above. In particular, let  $r \in [0, n]$ . Then there exists a sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to 0 such that*

(i) *if  $r \geq 1$ , then for all  $\nu \in P(r)$ ,*

$$\lim_{k \rightarrow \infty} \Delta_{1,r}^{KL}(\nu; \varepsilon_k) = (q; q)_\infty^{-2r} \Delta^L(\rho_L q^\nu; a, b; t) \prod_{i=1}^r \rho_{L,i} q^{\nu_i},$$

and there exists a  $K \in \mathbb{R}_{>0}$  independent of  $\nu \in P(r)$  such that

$$(3.7.4) \quad \sup_{k \in \mathbb{Z}_+} |\Delta_{1,r}^{KL}(\nu; \varepsilon_k)| \leq K \Delta^L(\rho_L q^\nu; a, b; t) \prod_{i=1}^r \rho_{L,i} q^{\nu_i}$$

for all  $\nu \in P(r)$ .

(iii) *if  $r < n$ , then  $\lim_{k \rightarrow \infty} \Delta_{2,r}^{KL}(\nu, z; \varepsilon_k) = 0$  for all  $z \in T^{n-r}$ ,  $\nu \in P(r)$  and*

$$\sup_{(\nu, z, k) \in P(r) \times T^{n-r} \times \mathbb{Z}_+} |\Delta_{2,r}^{KL}(\nu, z; \varepsilon_k)| < \infty.$$

Before a proof of Lemma 3.7.1 is given, we show how Lemma 3.7.1 implies Proposition 3.3.2. Since the infinite sum

$$(3.7.5) \quad (1-q)^{-n} \langle 1, 1 \rangle_{L,t}^{a,b} = \sum_{\nu \in P(n)} \Delta^L(\rho_L q^\nu; a, b; t) \prod_{i=1}^n \rho_{L,i} q^{\nu_i}$$

is absolutely convergent (cf. [116, proof of Proposition 6.1]) and

$$\sup_{\nu, \varepsilon, z} |m_\lambda(\rho_L q^\nu, \varepsilon q^{-\frac{1}{2}} z | \varepsilon q^{-\frac{1}{2}})| < \infty \quad (\lambda \in \Lambda),$$

where the supremum is taken over triples  $(\nu, \varepsilon, z)$  with  $\nu \in P_L(r; \varepsilon)$ ,  $\varepsilon \in \mathbb{R}_{>0}$  and  $z \in T^{n-r}$ , it follows by Lebesgue's dominated convergence Theorem, (3.3.9), (3.7.2), (3.7.3) and Lemma 3.7.1 that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left( \prod_{i=1}^n (-\varepsilon_k^{-1} q t^{i-1}, -\varepsilon_k^{-1} q a t^{i-1}; q)_\infty \right) (\varepsilon_k q^{-\frac{1}{2}})^{|\lambda|+|\mu|} \langle m_\lambda, m_\mu \rangle_{\underline{t}_L(\varepsilon_k), t} \\ &= \sum_{r, \nu} \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \iint_{z \in T^{n-r}} \lim_{k \rightarrow \infty} (m_\lambda m_\mu)(\rho_L q^\nu, \varepsilon_k q^{-\frac{1}{2}} z | \varepsilon_k q^{-\frac{1}{2}}) \mathcal{W}_r^L(\nu, z; \varepsilon_k) \frac{dz}{z} \\ &= 2^n n! (q; q)_\infty^{-2n} \sum_{\nu \in P(n)} (\tilde{m}_\lambda \tilde{m}_\mu)(\rho_L q^\nu) \Delta^L(\rho_L q^\nu; a, b; t) \prod_{i=1}^n \rho_{L,i} q^{\nu_i} \\ &= 2^n n! (1-q)^{-n} (q; q)_\infty^{-2n} \langle \tilde{m}_\lambda, \tilde{m}_\mu \rangle_{L,t}^{a,b} \end{aligned}$$

for some sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to 0, where the sum in the second line is over four tuples  $(r, \nu)$  with  $r \in [0, n]$ , and  $\nu \in P(r)$ . So for the proof of Proposition 3.3.2, it suffices to prove Lemma 3.7.1. We use the following elementary lemma.

LEMMA 3.7.2. *For given  $\varepsilon_0 \in \mathbb{R}_{>0}$ , set  $\varepsilon_k := \varepsilon_0 q^k$ .*

(a) *Let  $c \in \mathbb{C}$ . For  $\varepsilon_0 \in \mathbb{R}_{>0}$  with  $|c| \varepsilon_0 \notin \{q^{-l}\}_{l \in \mathbb{Z}_+}$  there exist positive constants  $K^\pm > 0$  which only depend on  $\varepsilon_0$  and  $|c|$ , such that  $K^- \leq |(c\varepsilon_k; q)_\infty| \leq K^+$  for all  $k \in \mathbb{Z}_+$ . Furthermore,  $\lim_{k \rightarrow \infty} (c\varepsilon_k; q)_\infty = 1$ .*

(b) *Let  $a, b \in \mathbb{C}^*$ , and set*

$$(3.7.6) \quad f_{\{l,m\}}(\varepsilon; a, b) := \frac{(\varepsilon^{-1} a q^{1-m}; q)_m}{(\varepsilon^{-1} b q^{1-m-l}; q)_m}, \quad (l, m \in \mathbb{Z}_+).$$

*Let  $\varepsilon_0 \in \mathbb{R}_{>0}$  such that  $\varepsilon_0^{-1} |b| \notin \{q^k\}_{k \in \mathbb{Z}_+}$ . Then there exists a positive constant  $K > 0$  which depends only on  $\varepsilon_0$ ,  $|a|$  and  $|b|$ , such that  $|f_{\{l,m\}}(\varepsilon_k; a, b)| \leq K |q^l a/b|^m$  for all  $k, l, m \in \mathbb{Z}_+$ . Furthermore,  $\lim_{k \rightarrow \infty} f_{\{l,m\}}(\varepsilon_k; a, b) = (q^l a/b)^m$ .*

(c) *Let  $u_i, v_j \in \mathbb{C}^*$  for  $i \in [1, r]$ ,  $j \in [1, s]$  and assume that  $r < s$ , or that  $r = s$  and  $|u_1 \dots u_r| < |v_1 \dots v_r|$ . Set*

$$(3.7.7) \quad g(\varepsilon) := \frac{(\varepsilon^{-1} u_1, \dots, \varepsilon^{-1} u_r; q)_\infty}{(\varepsilon^{-1} v_1, \dots, \varepsilon^{-1} v_s; q)_\infty}.$$

Let  $\varepsilon_0 \in \mathbb{R}_{>0}$  such that  $\varepsilon_0^{-1}|v_j| \notin \{q^l\}_{l \in \mathbb{Z}}$  for  $j \in [1, s]$ . Then there exists a positive constant  $K > 0$  which depends only on  $\varepsilon_0$ ,  $|u_i|$  and  $|v_j|$ , such that  $\sup_{k \in \mathbb{Z}_+} |g(\varepsilon_k)| \leq K$ . Furthermore,  $\lim_{k \rightarrow \infty} g(\varepsilon_k) = 0$ .

PROOF. The proof of **(a)** is straightforward. For **(b)** and **(c)** use (2.5.3) to rewrite  $f_{\{l, m\}}$  as

$$f_{\{l, m\}}(\varepsilon; a, b) = (q^l a/b)^m \frac{(a^{-1}\varepsilon; q)_m}{(b^{-1}q^l \varepsilon; q)_m},$$

and to rewrite  $g(\varepsilon_k)$  as

$$(3.7.8) \quad g(\varepsilon_k) = \left( \frac{u_1 \cdots u_r}{v_1 \cdots v_s} (-q^{(k+1)/2} \varepsilon_0)^{s-r} \right)^k \frac{(q\varepsilon_0 u_1^{-1}, \dots, q\varepsilon_0 u_r^{-1}; q)_k}{(q\varepsilon_0 v_1^{-1}, \dots, q\varepsilon_0 v_s^{-1}; q)_k} g(\varepsilon_0).$$

The limits for  $f_{\{l, m\}}$  and  $g$  given in **(b)** respectively **(c)** are now immediately clear. Furthermore we have the estimate  $|f_{\{l, m\}}(\varepsilon_k; a, b)| \leq K|q^l a/b|^m$  with

$$K = \frac{(-|a|^{-1}\varepsilon_0; q)_\infty}{(|b|^{-1}\varepsilon_0 q^{k_0}; q)_\infty} \prod_{\{i \in \mathbb{Z}_+ \mid 1 < |b|^{-1}\varepsilon_0 q^i < 2\}} (|b|^{-1}\varepsilon_0 q^i - 1)^{-1} > 0,$$

where  $k_0$  is the smallest positive integer such that  $|b|^{-1}\varepsilon_0 q^{k_0} < 1$ . The estimate for  $|g(\varepsilon_k)|$  in **(c)** is easily derived from (3.7.8), the assumptions on  $r, s$  and on the parameters  $u_i, v_j$ , and from estimates similar to the estimate for  $K$  in the proof of **(b)**.  $\square$

We proceed with the proof of Lemma 3.7.1. Set  $\varepsilon_k := \varepsilon_0 q^k$  for given  $\varepsilon_0 \in \mathbb{R}_{>0}$ .

PROOF OF LEMMA 3.7.1(i). By (2.4.7) we have

$$(3.7.9) \quad \Delta_{1,r}^{KL}(\nu; \varepsilon) = \delta_d(\rho_L(\varepsilon)q^\nu) \prod_{i=1}^r \{(-\varepsilon^{-1}qt^{i-1}, -\varepsilon^{-1}qat^{i-1}; q)_\infty \\ \cdot w_d(\rho_{L,i}(\varepsilon)q^{\nu_i}; \rho_{L,i}(\varepsilon)q^{\nu_i-1})\}$$

with  $\delta_d$  given by (2.4.8) and  $w_d$  given by (2.2.7). By (2.4.8) and (3.2.1), we have

$$(3.7.10) \quad \delta_d(\rho_L(\varepsilon)q^\nu) = F_1(\nu)G_1(\nu; \varepsilon)$$

with

$$(3.7.11) \quad F_1(\nu) := \prod_{1 \leq i < j \leq r} \frac{(t^{j-i}q^{\nu_j - \nu_i}; q)_\tau}{(t^{i-j}q^{\nu_{i-1} - \nu_j}; q)_{\nu_i - \nu_{i-1}}}, \\ G_1(\nu; \varepsilon) := \prod_{1 \leq i < j \leq r} \frac{(\varepsilon^2 t^{2-i-j} q^{-\nu_i - \nu_j - 1}; q)_\tau}{(\varepsilon^{-2} t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1}; q)_{\nu_i - \nu_{i-1}}},$$

for  $\nu \in P(r)$ , where  $\nu_0 = 0$  and  $t = q^\tau$ . By applying (2.5.3) to the  $q$ -shifted factorials in the denominator of  $F_1$  and using the formula

$$(3.7.12) \quad \sum_{i=1}^r (i-1)(r-i) = \binom{r}{3},$$

we obtain

$$(3.7.13) \quad F_1(\nu) = \delta_{qJ}(\rho_L q^\nu) q^{-2\tau^2 \binom{r}{3}} \prod_{j=1}^r \frac{(t^{j-1} q^{\nu_j+1}; q)_\infty}{(q^{\nu_j - \nu_{j-1} + 1}; q)_\infty} t^{-2(r-j)\nu_j} \prod_{1 \leq i < j \leq r} (-q^{\nu_j - \nu_i + 1} t^{j-i})^{\nu_i - \nu_{i-1}} q^{\binom{\nu_i - \nu_{i-1}}{2}},$$

where  $\delta_{qJ}$  (3.3.6) is the interaction factor for the weight function of the little  $q$ -Jacobi polynomials. On the other hand, we have for  $i \in [1, r]$  by (2.2.7),

$$(3.7.14) \quad (-\varepsilon^{-1} q t^{i-1}, -\varepsilon^{-1} q a t^{i-1}; q)_\infty w_d(\rho_{L,i}(\varepsilon) q^{\nu_i}; \rho_{L,i}(\varepsilon) q^{\nu_{i-1}}) = I_{1,i}(\nu) J_{1,i}(\nu; \varepsilon)$$

with

$$\begin{aligned} I_{1,i}(\nu) &:= \frac{(t^{i-1} q^{\nu_{i-1}+1} a b)^{\nu_{i-1} - \nu_i}}{(q, q b t^{i-1} q^{\nu_i}; q)_\infty (q; q)_{\nu_i - \nu_{i-1}}} \\ &= v_L(\rho_{L,i} q^{\nu_i}) \frac{a^{(1-i)\tau - \nu_i} (t^{i-1} q^{\nu_{i-1}+1} a b)^{\nu_{i-1} - \nu_i}}{(q, t^{i-1} q^{\nu_i+1}; q)_\infty (q; q)_{\nu_i - \nu_{i-1}}} \end{aligned}$$

(here  $v_L$  (3.2.13) is the one variable weight function of the little  $q$ -Jacobi polynomials) and with  $J_{1,i}(\nu; \varepsilon)$  given by

$$\begin{aligned} J_{1,i}(\nu; \varepsilon) &:= \frac{(\varepsilon^2 t^{2-2i} q^{-2\nu_{i-1}-1}; q)_\infty}{(\varepsilon^2 b t^{1-i} q^{-\nu_{i-1}}, -\varepsilon t^{1-i} q^{-\nu_{i-1}}, -\varepsilon a t^{1-i} q^{-\nu_{i-1}}; q)_\infty} \\ &\quad \frac{(\varepsilon^{-2} q^{1+2\nu_{i-1}} t^{2i-2}; q)_{\nu_i - \nu_{i-1}} (-\varepsilon^{-1} t^{i-1} q; q)_{\nu_{i-1}} (-\varepsilon^{-1} a t^{i-1} q; q)_{\nu_i}}{(\varepsilon^{-2} b^{-1} t^{i-1} q^{\nu_{i-1}+1}, -\varepsilon^{-1} a^{-1} t^{i-1} q^{\nu_{i-1}+1}; q)_{\nu_i - \nu_{i-1}}} \\ &\quad \frac{(1 - \varepsilon^{-2} q^{1+2\nu_i} t^{2i-2})}{(1 - \varepsilon^{-2} q^{1+2\nu_{i-1}} t^{2i-2})}. \end{aligned}$$



Thus by (3.7.9), (3.7.10), (3.7.13) and (3.7.14), we have  $\Delta_{1,r}^{KL}(\nu; \varepsilon) = M_1(\nu)N_1(\nu; \varepsilon)$  for  $\varepsilon \in \mathbb{R}_{>0}$  and  $\nu \in P_L(r; \varepsilon)$  with

$$(3.7.15) \quad \begin{aligned} M_1(\nu) &:= F_1(\nu) \prod_{i=1}^r I_{1,i}(\nu) \\ &= \Delta^L(\rho_L q^\nu)(q; q)_\infty^{-2r} t^{\binom{r}{2}} \prod_{i=1}^r (t^{-2(r-i)} a^{-1})^{\nu_i} (t^{i-1} q^{\nu_{i-1}+1} ab)^{\nu_{i-1}-\nu_i} \\ &\quad \cdot \prod_{1 \leq i < j \leq r} (-q^{\nu_j - \nu_i + 1} t^{j-i})^{\nu_i - \nu_{i-1}} q^{\binom{\nu_i - \nu_{i-1}}{2}}, \end{aligned}$$

( $\Delta^L$  given by (3.3.5)) and with

$$N_1(\nu; \varepsilon) := G_1(\nu; \varepsilon) \prod_{i=1}^r J_{1,i}(\nu; \varepsilon).$$

Now replace the factor  $(-\varepsilon^{-1} a t^{i-1} q; q)_{\nu_i}$  in  $J_{1,i}(\nu; \varepsilon)$  by

$$(-\varepsilon^{-1} a t^{i-1} q^{\nu_{i-1}+1}; q)_{\nu_i - \nu_{i-1}} (-\varepsilon^{-1} a t^{i-1} q; q)_{\nu_{i-1}}$$

for  $i \in [1, r]$ , then  $N_1(\nu; \varepsilon)$  can explicitly be given by

$$(3.7.16) \quad N_1(\nu; \varepsilon) = N_1^1(\nu; \varepsilon) N_1^2(\nu; \varepsilon) N_1^3(\nu; \varepsilon)$$

with

$$(3.7.17) \quad \begin{aligned} N_1^1(\nu; \varepsilon) &:= \prod_{i=1}^r \frac{(\varepsilon^2 t^{2-2i} q^{-2\nu_{i-1}-1}; q)_\infty}{(\varepsilon^2 b t^{1-i} q^{-\nu_{i-1}}, -\varepsilon t^{1-i} q^{-\nu_{i-1}}, -\varepsilon a t^{1-i} q^{-\nu_{i-1}}; q)_\infty} \\ &\quad \cdot \prod_{1 \leq i < j \leq r} (\varepsilon^2 t^{2-i-j} q^{-\nu_i - \nu_j - 1}; q)_\tau \end{aligned}$$

if  $\nu \in P_L(r; \varepsilon)$  and zero otherwise,

$$(3.7.18) \quad \begin{aligned} N_1^2(\nu; \varepsilon) &:= \prod_{i=1}^r \left( \frac{(\varepsilon^{-2} q^{1+2\nu_{i-1}} t^{2i-2}, -\varepsilon^{-1} a t^{i-1} q^{1+\nu_{i-1}}; q)_{\nu_i - \nu_{i-1}}}{(\varepsilon^{-2} b^{-1} t^{i-1} q^{1+\nu_{i-1}}, -\varepsilon^{-1} a^{-1} t^{i-1} q^{1+\nu_{i-1}}; q)_{\nu_i - \nu_{i-1}}} \right. \\ &\quad \left. \cdot \frac{(\varepsilon^{-2} q^{1+2\nu_i} t^{2i-2}; q)_1}{(\varepsilon^{-2} q^{1+2\nu_{i-1}} t^{2i-2}; q)_1} \right) \end{aligned}$$

if  $\nu \in P_L(r; \varepsilon)$  and zero otherwise, and

$$(3.7.19) \quad N_1^3(\nu; \varepsilon) = \frac{\prod_{i=1}^r (-\varepsilon^{-1} t^{i-1} q, -\varepsilon^{-1} a t^{i-1} q; q)_{\nu_{i-1}}}{\prod_{1 \leq i < j \leq r} (\varepsilon^{-2} t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1}; q)_{\nu_i - \nu_{i-1}}}$$

if  $\nu \in P_L(r; \varepsilon)$  and zero otherwise. For the factor  $N_1^1$  we have for generic  $\varepsilon_0 > 0$ ,

$$(3.7.20) \quad \lim_{k \rightarrow \infty} N_1^1(\nu; \varepsilon_k) = 1 \quad (\nu \in P(r))$$

by Lemma 3.7.2(a). Lemma 3.7.2(b) can be used to calculate the limit of the factor  $N_1^2$ . We obtain for generic  $\varepsilon_0 > 0$ ,

$$(3.7.21) \quad \lim_{k \rightarrow \infty} N_1^2(\nu; \varepsilon_k) = \prod_{i=1}^r (q^{\nu_{i-1}+2} t^{i-1} a^2 b)^{\nu_i - \nu_{i-1}}$$

for all  $\nu \in P(r)$ . As an example, the limit of a factor of  $N_1^2$  will be calculated explicitly using Lemma 3.7.2(b). Consider the factor

$$(3.7.22) \quad N_1^{i,2}(\nu; \varepsilon) := \frac{(\varepsilon^{-2} q^{1+2\nu_{i-1}} t^{2i-2}; q)_{\nu_i - \nu_{i-1}}}{(\varepsilon^{-2} q^{1+\nu_{i-1}} b^{-1} t^{i-1}; q)_{\nu_i - \nu_{i-1}}}$$

of  $N_1^2(\nu; \varepsilon)$  for some  $i \in [1, r]$ . Then for generic  $\varepsilon_0 > 0$ , it follows from Lemma 3.7.2(b) that

$$(3.7.23) \quad \begin{aligned} \lim_{k \rightarrow \infty} N_1^{i,2}(\nu; \varepsilon_k) &= \lim_{k \rightarrow \infty} N_1^{i,2}(\nu; \varepsilon_{k+\nu_r}) \\ &= \lim_{k \rightarrow \infty} N_1^{i,2}(\nu; q^{\nu_r} \varepsilon_k) \\ &= \lim_{k \rightarrow \infty} f_{\{\nu_{i-1}, \nu_i - \nu_{i-1}\}}(\varepsilon_{2\nu_r - \nu_i - \nu_{i-1} + 2k}; \varepsilon_0^{-1} t^{2i-2}, \varepsilon_0^{-1} b^{-1} t^{i-1}) \\ &= (q^{\nu_{i-1}} t^{i-1} b)^{\nu_i - \nu_{i-1}}. \end{aligned}$$

Similarly, the limits of the other factors of  $N_1^2$  can be computed, which yield (3.7.21). Finally, we have for generic  $\varepsilon_0 > 0$ ,

$$(3.7.24) \quad \begin{aligned} \lim_{k \rightarrow \infty} N_1^3(\nu; \varepsilon_k) &= \prod_{i=1}^r (at^{2(i-1)} q^{\nu_{i-1}+1})^{\nu_{i-1}} \\ &\quad \cdot \prod_{1 \leq i < j \leq r} (-t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1})^{\nu_{i-1} - \nu_i} q^{-\binom{\nu_i - \nu_{i-1}}{2}} \end{aligned}$$

since

$$(3.7.25) \quad \sum_{i=1}^r \nu_{i-1} = \sum_{1 \leq i < j \leq r} (\nu_i - \nu_{i-1}) \quad (\nu \in P(r)).$$

We thus obtain for generic  $\varepsilon_0 > 0$  by (3.7.16), (3.7.20), (3.7.21) and (3.7.24),

$$(3.7.26) \quad \begin{aligned} \lim_{k \rightarrow \infty} N_1(\nu; \varepsilon_k) &= \prod_{i=1}^r t^{(i-1)(\nu_i + \nu_{i-1})} q^{\nu_{i-1} \nu_i - \nu_{i-1} + 2\nu_i} a^{2\nu_i - \nu_{i-1}} b^{\nu_i - \nu_{i-1}} \\ &\quad \cdot \prod_{1 \leq i < j \leq r} (-t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1})^{\nu_{i-1} - \nu_i} q^{-\binom{\nu_i - \nu_{i-1}}{2}} \end{aligned}$$

for all  $\nu \in P(r)$ . By (3.7.15) and (3.7.26) we obtain for generic  $\varepsilon_0 > 0$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} \Delta_{1,r}^{KL}(\nu; \varepsilon_k) &= M_1(\nu) \lim_{k \rightarrow \infty} N_1(\nu; \varepsilon_k) \\ &= (q; q)_\infty^{-2r} \Delta^L(\rho_L q^\nu) t^{\binom{r}{2}} q^{|\nu|} \\ &= (q; q)_\infty^{-2r} \Delta^L(\rho_L q^\nu) \prod_{i=1}^r \rho_{L,i} q^{\nu_i} \end{aligned}$$

for all  $\nu \in P(r)$ , where  $\Delta^L$  is given by (3.3.5) and  $|\nu| := \nu_1 + \cdots + \nu_r$  for  $\nu \in P(r)$ .

To prove the estimate (3.7.4), we use the estimates of Lemma 3.7.2 (a) and (b) for (factors of)  $N_1$ . For  $N_1^1$ , we use Lemma 3.7.2(a) and the condition that  $N_1^1(\nu; \varepsilon) = 0$  if  $\nu \notin P_L(r; \varepsilon)$  to prove that  $\sup_{\nu, k} |N_1^1(\nu; \varepsilon_k)| < \infty$  for generic  $\varepsilon_0 > q^{\frac{1}{2}}$ . Indeed, since  $\nu \in P_L(r; \varepsilon)$  implies  $\varepsilon < q^{\frac{1}{2} + \nu_r}$ , we have for  $\varepsilon_0 > q^{\frac{1}{2}}$ ,

$$(3.7.27) \quad \sup_{(\nu, k) \in P(r) \times \mathbb{Z}_+} |N_1^1(\nu; \varepsilon_k)| = \sup_{(\nu, k) \in P(r) \times \mathbb{Z}_+} |N_1^1(\nu; \varepsilon_k q^{\nu_r})| < \infty$$

by (3.7.17) and Lemma 3.7.2(a).

For  $N_1^2(\nu; \varepsilon)$  we want to establish the estimate

$$(3.7.28) \quad \sup_{k \in \mathbb{Z}_+} |N_1^2(\nu; \varepsilon_k)| \leq K_1^2 \prod_{i=1}^r (q^{\nu_{i-1} + 2} t^{i-1} a^2 |b|)^{\nu_i - \nu_{i-1}}$$

for generic  $\varepsilon_0 > q^{\frac{1}{2}}$  with  $K_1^2 > 0$  independent of  $\nu \in P(r)$ , in view of the limit (3.7.21). This can be done with the help of Lemma 3.7.2(b). As an example, consider the factor  $N_1^{i,2}(\nu; \varepsilon)$  (3.7.22). In view of the limit (3.7.23), we want to prove the estimate

$$\sup_{k \in \mathbb{Z}_+} |N_1^{i,2}(\nu; \varepsilon_k)| \leq K_1^{i,2} (q^{\nu_{i-1}} t^{i-1} |b|)^{\nu_i - \nu_{i-1}}$$

for generic  $\varepsilon_0 > q^{\frac{1}{2}}$  with  $K_1^{i,2} > 0$  independent of  $\nu \in P(r)$ . This follows for generic  $\varepsilon_0 > q^{\frac{1}{2}}$ , using the fact that  $N_1^{i,2}(\nu; \varepsilon) = 0$  if  $\nu \notin P_L(r; \varepsilon)$ , by the estimates

$$\begin{aligned} \sup_{k \in \mathbb{Z}_+} |N_1^{i,2}(\nu; \varepsilon_k)| &= \sup_{k \in \mathbb{Z}_+} |N_1^{i,2}(\nu; q^{\nu_r} \varepsilon_k)| \\ &= \sup_{k \in \mathbb{Z}_+} |f_{\{\nu_{i-1}, \nu_i - \nu_{i-1}\}}(\varepsilon_{2\nu_r - \nu_i - \nu_{i-1} + 2k}; \varepsilon_0^{-1} t^{2i-2}, \varepsilon_0^{-1} b^{-1} t^{i-1})| \\ &\leq \sup_{k \in \mathbb{Z}_+} |f_{\{\nu_{i-1}, \nu_i - \nu_{i-1}\}}(\varepsilon_k; \varepsilon_0^{-1} t^{2i-2}, \varepsilon_0^{-1} b^{-1} t^{i-1})| \\ &\leq K_1^{i,2} (q^{\nu_{i-1}} t^{i-1} |b|)^{\nu_i - \nu_{i-1}} \end{aligned}$$

with  $K_1^{i,2}$  independent of  $\nu$  by Lemma 3.7.2(b) (cf. (3.7.23)). In a similar manner, estimates can be given for the other factors of  $N_1^2$ .

For  $N_1^3$ , we want to prove that

$$(3.7.29) \quad \sup_{k \in \mathbb{Z}_+} |N_1^3(\nu; \varepsilon_k)| \leq K_1^3 \prod_{i=1}^r (at^{2(i-1)}q^{\nu_{i-1}+1})^{\nu_{i-1}} \cdot \prod_{1 \leq i < j \leq r} (t^{i+j-2}q^{\nu_{i-1}+\nu_j+1})^{\nu_{i-1}-\nu_i} q^{-(\nu_i-\frac{\nu_{i-1}}{2})}$$

for generic  $\varepsilon_0 > q^{\frac{1}{2}}$  with  $K_1^3 > 0$  independent of  $\nu \in P(r)$ , in view of the limit (3.7.24). This follows by straightforward estimates, using (2.5.3) and the fact that  $N_1^3(\nu; \varepsilon) = 0$  if  $\nu \notin P_L(r; \varepsilon)$ . Hence by (3.7.16), (3.7.27), (3.7.28) and (3.7.29) we have the estimate

$$(3.7.30) \quad \sup_{k \in \mathbb{Z}_+} |N_1(\nu; \varepsilon_k)| \leq K_1 \prod_{i=1}^r t^{(i-1)(\nu_i+\nu_{i-1})} q^{\nu_{i-1}\nu_i-\nu_{i-1}+2\nu_i} a^{2\nu_i-\nu_{i-1}} |b|^{\nu_i-\nu_{i-1}} \cdot \prod_{1 \leq i < j \leq r} (t^{i+j-2}q^{\nu_{i-1}+\nu_j+1})^{\nu_{i-1}-\nu_i} q^{-(\nu_i-\frac{\nu_{i-1}}{2})}$$

for generic  $\varepsilon_0 > q^{\frac{1}{2}}$ , with  $K_1 > 0$  independent of  $\nu \in P(r)$ , so in particular,

$$\sup_{k \in \mathbb{Z}_+} |\Delta_{1,r}^{KL}(\nu; \varepsilon_k)| = |M_1(\nu)| \sup_{k \in \mathbb{Z}_+} |N_1(\nu; \varepsilon_k)| \leq K \Delta^L(\rho_L q^\nu) \prod_{i=1}^r \rho_{L,i} q^{\nu_i}$$

with  $K > 0$  independent of  $\nu \in P(r)$ . This completes the proof of Lemma 3.7.1(i).  $\square$

**PROOF OF LEMMA 3.7.1(ii).** Using the explicit formulas for the weight function  $\Delta$  (2.3.4), (2.2.3), (2.3.5) and for  $\delta_c$  (2.4.11) as well as the definition of  $t_L(\varepsilon)$  (3.2.1), we can rewrite  $\Delta_{2,r}^{KL}(\nu, z; \varepsilon)$  as

$$\Delta_{2,r}^{KL}(\nu, z; \varepsilon) = N_2^1(z) N_2^2(\nu, z; \varepsilon) N_2^3(z; \varepsilon) N_2^4(\nu, z; \varepsilon)$$

with

$$(3.7.31) \quad N_2^1(z) := \delta(z; t) \prod_{i=1}^{n-r} \frac{(z_i^2, z_i^{-2}; q)_\infty}{(-q^{\frac{1}{2}}z_i, -q^{\frac{1}{2}}z_i^{-1}, -q^{\frac{1}{2}}az_i, -q^{\frac{1}{2}}az_i^{-1}; q)_\infty}$$

where  $\delta$  is given by (2.3.5),

$$(3.7.32) \quad N_2^2(\nu, z; \varepsilon) := \prod_{j=1}^{n-r} \frac{\prod_{i=1}^r (\varepsilon q^{-\frac{1}{2}-\nu_i} t^{1-i} z_j, \varepsilon q^{-\frac{1}{2}-\nu_i} t^{1-i} z_j^{-1}; q)_r}{(\varepsilon b q^{\frac{1}{2}} z_j, \varepsilon b q^{\frac{1}{2}} z_j^{-1}; q)_\infty} \quad (t = q^r)$$

if  $\nu \in P_L(r; \varepsilon)$  and zero otherwise,

$$(3.7.33) \quad N_2^3(z; \varepsilon) := \prod_{i=1}^{n-r} \frac{(-\varepsilon^{-1} q t^{r+i-1}, -\varepsilon^{-1} q a t^{r+i-1}, \varepsilon^{-1} q^{\frac{1}{2}} z_i, \varepsilon^{-1} q^{\frac{1}{2}} z_i^{-1}; q)_\infty}{(\varepsilon^{-1} q^{\frac{1}{2}} z_i, \varepsilon^{-1} q^{\frac{1}{2}} z_i^{-1}, \varepsilon^{-1} q^{\frac{1}{2}} t^r z_i, \varepsilon^{-1} q^{\frac{1}{2}} t^r z_i^{-1}; q)_\infty}$$

and

$$(3.7.34) \quad N_2^4(\nu, z; \varepsilon) := \prod_{\substack{0 \leq i \leq r-1 \\ 1 \leq j \leq n-r}} \frac{(\varepsilon^{-1} q^{\frac{1}{2} + \nu_i} t^r z_j, \varepsilon^{-1} q^{\frac{1}{2} + \nu_i} t^r z_j^{-1}; q)_{\nu_{i+1} - \nu_i}}{(\varepsilon^{-1} q^{\frac{1}{2} + \nu_i} t^i z_j, \varepsilon^{-1} q^{\frac{1}{2} + \nu_i} t^i z_j^{-1}; q)_{\nu_{i+1} - \nu_i}}$$

if  $\nu \in P_L(r; \varepsilon)$  and zero otherwise, where  $\nu_0 = 0$ . Note that  $N_3^1$  is bounded on  $T^{n-r}$ . For  $N_2^2$  it follows from Lemma 3.7.2 (a) that  $\lim_{k \rightarrow \infty} N_2^2(\nu, z; \varepsilon_k) = 1$  for all  $\nu \in P(r)$ ,  $z \in T^{n-r}$  and that

$$\sup_{(\nu, z, k) \in P(r) \times T^{n-r} \times \mathbb{Z}_+} |N_2^2(\nu, z; \varepsilon_k)| < \infty$$

for generic  $\varepsilon_0 > q^{\frac{1}{2}}$ . By Lemma 3.7.2(c) and the fact that  $0 < a < 1/q$ , we have for generic  $\varepsilon_0 > 0$  that  $\lim_{k \rightarrow \infty} N_2^3(z; \varepsilon_k) = 0$  for all  $z \in T^{n-r}$  and

$$\sup_{(z, k) \in T^{n-r} \times \mathbb{Z}_+} |N_2^3(z; \varepsilon_k)| < \infty.$$

Finally, Lemma 3.7.2(b) can be used to prove that  $\lim_{k \rightarrow \infty} N_3^4(\nu, z; \varepsilon_k) = t^{2(n-r)|\nu|}$  for all  $\nu \in P(r)$ ,  $z \in T^{n-r}$  and that

$$\sup_{(\nu, z, k) \in P(r) \times T^{n-r} \times \mathbb{Z}_+} |N_2^4(\nu, z; \varepsilon_k)| < \infty$$

for generic  $\varepsilon_0 > q^{\frac{1}{2}}$ . This completes the proof of Lemma 3.7.1(ii).  $\square$

### 3.8. Limit of the orthogonality measure (big $q$ -Jacobi case)

In the next lemma a new expression is given for the weight  $\underline{c}_B$  which appears in the definition of  $\langle \cdot, \cdot \rangle_B$ .

LEMMA 3.8.1. *The weight  $\underline{c}_B \in (\mathbb{C}^*)^{n+1}$  can be rewritten as*

$$(3.8.1) \quad \begin{aligned} c_{B,j} &= (q; q)_\infty^n \prod_{i=1}^j \frac{\theta(-t^{i+j-n}c/d)}{\theta(-t^{1-i}d/c)\theta(-t^i c/d)} \prod_{i=1}^{n-j} \frac{1}{\theta(-t^{1-i}c/d)} \\ &\cdot q^{-2\tau^2 \left( (n-j) \binom{j}{2} + \binom{j}{3} + \binom{n-j}{3} \right)} t^{-\binom{j}{2} - \binom{n-j}{2}} \\ &\cdot c^{-2\tau \left( j(n-j) + \binom{j}{2} \right) - j} d^{-2\tau \binom{n-j}{2} + j - n} \end{aligned}$$

for  $j \in [0, n]$ .

PROOF. For  $j = 0$ , (3.8.1) follows from (3.4.7) since  $c_{B,j} = c_B d_{B,j}$  with  $d_{B,0} = 1$ . For  $j \in [1, n]$ , write  $\tilde{c}_{B,j}$  for the right hand side of (3.8.1), then by the explicit expression (3.4.4) for  $d_{B,j}$  it remains to prove that

$$\frac{\tilde{c}_{B,j}}{\tilde{c}_{B,j-1}} = \prod_{m=j+1}^n \Psi_t(-t^{n-m-j+1}d/c)$$

for  $j \in [1, n]$ , with  $\Psi_t$  given by (3.4.5). This follows by a direct calculation.  $\square$

The remainder of this section is devoted to a proof of Proposition 3.4.4.

Fix in this section  $(a, b, c, d) \in V_B$  with  $a, b \neq 0$ . With slight modifications of the formulas, the proof goes also through for  $a = 0$  or  $b = 0$ .

For  $\varepsilon \in \mathbb{R}_{>0}$  we write  $\rho_{B,j}(\varepsilon) := t_{B,0}(\varepsilon)t^{j-1} = (qc/d)^{\frac{1}{2}}\varepsilon^{-1}t^{j-1}$  and we write  $\sigma_{B,j}(\varepsilon) := t_{B,1}(\varepsilon)t^{j-1} = -(qd/c)^{\frac{1}{2}}\varepsilon^{-1}t^{j-1}$  for  $j \in \mathbb{Z}$ , where  $t_B(\varepsilon)$  is given by (3.2.10). Then for  $\varepsilon > 0$  sufficiently small, we have

$$F(r; \underline{t}_B(\varepsilon); t) = \bigcup_{\substack{l+m=r \\ l, m \in \mathbb{Z}_+}} D_0(l; \underline{t}_B(\varepsilon); t) \times D_1(m; \underline{t}_B(\varepsilon); t) \subset \mathbb{C}^r$$

where  $F(r)$  is given by (2.6.3) and

$$(3.8.2) \quad \begin{aligned} D_0(l; \underline{t}_B(\varepsilon); t) &= \{\rho_B(\varepsilon)q^\nu \mid \nu \in P_B^{(0)}(l; \varepsilon)\}, \\ P_B^{(0)}(l; \varepsilon) &:= \{\nu \in P(l) \mid |\rho_{B,l}(\varepsilon)q^{\nu_l}| > 1\}, \end{aligned}$$

if  $l > 0$ , respectively

$$(3.8.3) \quad \begin{aligned} D_1(m; \underline{t}_B(\varepsilon); t) &= \{\sigma_B(\varepsilon)q^\nu \mid \nu \in P_B^{(1)}(m; \varepsilon)\}, \\ P_B^{(1)}(m; \varepsilon) &:= \{\nu \in P(m) \mid |\sigma_{B,m}(\varepsilon)q^{\nu_m}| > 1\} \end{aligned}$$

if  $m > 0$ . Write

$$(3.8.4) \quad \begin{aligned} &\left( \prod_{i=1}^n (-\varepsilon^{-2}qt^{i-1}; q)_\infty \right) ((cd/q)^{\frac{1}{2}}\varepsilon)^{|\lambda|+|\mu|} \langle m_\lambda, m_\mu \rangle_{\underline{t}_B(\varepsilon), t} \\ &= \sum_{r, l, m, \nu, \nu'} \iint_{z \in T^{n-r}} (m_\lambda m_\mu) (\rho_B q^\nu, \sigma_B q^{\nu'}, (cd/q)^{\frac{1}{2}}\varepsilon z | (cd/q)^{\frac{1}{2}}\varepsilon) \mathcal{W}_{l, m, r}^B(\nu, \nu', z; \varepsilon) \frac{dz}{z} \end{aligned}$$

where  $\rho_{B,i} := ct^{i-1}$ ,  $\sigma_{B,i} := -dt^{i-1}$  and  $m_\lambda(z|u)$  is given by (3.3.8), and the sum is over five tuples  $(r, l, m, \nu, \nu')$  with  $r \in [0, n]$ ,  $l, m \in \mathbb{Z}_+$  with  $l + m = r$ ,  $\nu \in P(l)$ ,  $\nu' \in P(m)$ , and with the renormalized weight  $\mathcal{W}_{l, m, r}^B(\nu, \nu', z; \varepsilon)$  given by

$$(3.8.5) \quad \mathcal{W}_{l, m, r}^B(\nu, \nu', z; \varepsilon) := \frac{2^r (n-r+1)_r}{(2\pi i)^{n-r}} \Delta_{1, l, m}^{KB}(\nu, \nu'; \varepsilon) \Delta_{2, l, m}^{KB}(\nu, \nu', z; \varepsilon)$$

when  $r = l + m$ , with

$$(3.8.6) \quad \begin{aligned} \Delta_{1, l, m}^{KB}(\nu, \nu'; \varepsilon) &:= \left( \prod_{i=1}^r (-\varepsilon^{-2}qt^{i-1}; q)_\infty \right) \delta_c(\rho_B(\varepsilon)q^\nu; \sigma_B(\varepsilon)q^{\nu'}) \\ &\quad \cdot \Delta^{(d)}(\rho_B(\varepsilon)q^\nu; t_{B,0}(\varepsilon)) \Delta^{(d)}(\sigma_B(\varepsilon)q^{\nu'}; t_{B,1}(\varepsilon)) \end{aligned}$$

if  $\nu \in P_B^{(0)}(l; \varepsilon)$ ,  $\nu' \in P_B^{(1)}(m; \varepsilon)$  and zero otherwise,

$$(3.8.7) \quad \Delta_{2,l,m}^{KB}(\nu, \nu', z; \varepsilon) := \prod_{i=1}^{n-r} (-\varepsilon^{-2} q t^{r+i-1}; q)_{\infty} \\ \cdot \Delta(z; \underline{t}_B(\varepsilon); t) \delta_c(\rho_B(\varepsilon) q^{\nu}; z) \delta_c(\sigma_B(\varepsilon) q^{\nu'}; z)$$

if  $\nu \in P_B^{(0)}(l; \varepsilon)$ ,  $\nu' \in P_B^{(1)}(m; \varepsilon)$  and zero otherwise, with  $\delta_c$  given by (2.4.11). The obvious conventions are used when  $l = 0$ ,  $m = 0$  or  $r = n$  (compare with the little  $q$ -Jacobi case in Section 3.7). In particular,  $\Delta_{2,l,n-l}^{KB}(\nu, \nu'; \varepsilon) = 1$  for  $\nu \in P_B^{(0)}(l; \varepsilon)$ ,  $\nu' \in P_B^{(1)}(n-l; \varepsilon)$  and  $l \in [0, n]$ .

The following lemma will be used to pull a limit  $\varepsilon_k \downarrow 0$  in the right hand side of (3.8.4) through the integration over  $z \in T^{n-r}$  and through the infinite sums over  $\nu \in P(l)$  and  $\nu' \in P(m)$  for some sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to 0.

**LEMMA 3.8.2.** *Keep the notations and conventions as above. Let  $l, m \in \mathbb{Z}_+$  with  $l + m \in [0, n]$  and write  $r := l + m$ . Then there exists a sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to 0 such that*

(i) *for all  $\nu \in P(l)$ ,  $\nu' \in P(m)$*

$$\lim_{k \rightarrow \infty} \Delta_{1,l,m}^{KB}(\nu, \nu'; \varepsilon_k) = (q; q)_{\infty}^{-2r} c_{B,l} \Delta^B(\rho_B q^{\nu}, \sigma_B q^{\nu'}) \prod_{i=1}^l \rho_{B,i} q^{\nu_i} \prod_{j=1}^m |\sigma_{B,j}| q^{\nu'_j},$$

and there exists a  $K \in \mathbb{R}_{>0}$  independent of  $\nu \in P(l)$  and  $\nu' \in P(m)$  such that

$$\sup_{k \in \mathbb{Z}_+} |\Delta_{1,l,m}^{KB}(\nu, \nu'; \varepsilon_k)| \leq K c_{B,l} \Delta^B(\rho_B q^{\nu}, \sigma_B q^{\nu'}) \prod_{i=1}^l \rho_{B,i} q^{\nu_i} \prod_{j=1}^m |\sigma_{B,j}| q^{\nu'_j}.$$

for all  $\nu \in P(l)$  and all  $\nu' \in P(m)$ , where  $\Delta^B(z) = \Delta^B(z; a, b, c, d; t)$  is given by (3.4.3).

(ii) *if  $r < n$ , then  $\lim_{k \rightarrow \infty} \Delta_{2,l,m}^{KB}(\nu, \nu', z; \varepsilon_k) = 0$  for all  $\nu \in P(l)$ ,  $\nu' \in P(m)$ ,  $z \in T^{n-r}$  and*

$$\sup_{(k, \nu, \nu', z) \in \mathbb{Z}_+ \times P(l) \times P(m) \times T^{n-r}} |\Delta_{2,l,m}^{KB}(\nu, \nu', z; \varepsilon_k)| < \infty.$$

The proof of Proposition 3.4.4 is now an easy consequence of Lemma 3.8.2 and Lebesgue's dominated convergence Theorem. Indeed, the infinite sum

$$(3.8.8) \quad (1-q)^{-n} \langle 1, 1 \rangle_B = \sum_{(\nu, \nu', l)} c_{B,l} \Delta^B(\rho_B q^{\nu}, \sigma_B q^{\nu'}) \prod_{i=1}^l \rho_{B,i} q^{\nu_i} \prod_{j=1}^{n-l} |\sigma_{B,j}| q^{\nu'_j}$$

is absolutely convergent (cf. [116, proof of Proposition 6.1]), where the sum is taken over the three tuples  $(\nu, \nu', l)$  with  $\nu \in P(l)$  and  $\nu' \in P(n-l)$  and  $l \in [0, n]$ . Furthermore,

$$\sup_{\nu, \nu', z, \varepsilon} |m_{\lambda}(\rho_B q^{\nu}, \sigma_B q^{\nu'}, (cd/q)^{\frac{1}{2}} \varepsilon z | (cd/q)^{\frac{1}{2}} \varepsilon)| < \infty,$$

where the supremum is taken over the four tuples  $(\nu, \nu', z, \varepsilon)$  with  $\nu \in P_B^{(0)}(l; \varepsilon)$ ,  $\nu' \in P_B^{(1)}(m; \varepsilon)$ ,  $z \in T^{n-r}$  ( $r = l + m$ ) and  $\varepsilon > 0$ , hence it follows from Lebesgue's dominated convergence Theorem, (3.3.9), (3.8.4), (3.8.8) and Lemma 3.8.2 that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \left( \prod_{i=1}^n (-\varepsilon_k^{-2} q t^{i-1}; q)_\infty \right) ((cd/q)^{\frac{1}{2}} \varepsilon_k)^{|\lambda|+|\mu|} \langle m_\lambda, m_\mu \rangle_{\underline{t}_B(\varepsilon_k), t} \\ &= \frac{2^n n!}{(q; q)_\infty^{2n}} \sum_{l=0}^n \sum_{\substack{\nu \in P(l) \\ \nu' \in P(n-l)}} c_{B,l}(\tilde{m}_\lambda \tilde{m}_\mu \Delta^B) (\rho_B q^\nu, \sigma_B q^{\nu'}) \prod_{i=1}^l \rho_{B,i} q^{\nu_i} \prod_{j=1}^{n-l} |\sigma_{B,j}| q^{\nu'_j} \\ &= 2^n n! (1-q)^{-n} (q; q)_\infty^{-2n} \langle \tilde{m}_\lambda, \tilde{m}_\mu \rangle_{B,t}^{a,b,c,d}, \end{aligned}$$

for some sequence  $\{\varepsilon_k\}_{k \in \mathbb{Z}_+}$  in  $\mathbb{R}_{>0}$  converging to 0. So for the proof of Proposition 3.4.4, it suffices to prove Lemma 3.8.2.

*Proof of Lemma 3.8.2.*

Using the explicit expressions for  $\Delta^{(d)}$  (2.4.7),  $\delta_c$  (2.4.11) and  $\underline{t}_B(\varepsilon)$  (3.2.10), we can write

$$(3.8.9) \quad \Delta_{1,l,m}^{KB}(\nu, \nu'; \varepsilon) := U_0(\nu, \nu'; l, m) U_+(\varepsilon; \nu, \nu'; l, m) U_-(\varepsilon; \nu, \nu'; l, m)$$

with  $U_0, U_+$  respectively  $U_-$  the factor of  $\Delta_{1,l,m}^K$  consisting of products of  $q$ -shifted factorials of the form  $(e; q)_s$ ,  $(\varepsilon^2 e; q)_s$  respectively  $(\varepsilon^{-2} e; q)_s$  ( $s \in \mathbb{Z}_+ \cup \{\infty\}$ ). By a straightforward computation, the factors  $U_0, U_+$  and  $U_-$  can be explicitly given by

$$(3.8.10) \quad \begin{aligned} U_0(\nu, \nu'; l, m) &:= \Psi_0(\nu, l; a, b, c, d) \Psi_0(\nu', m; b, a, d, c) \\ &\cdot \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} (-t^{i-j} q^{\nu_i - \nu'_j} c/d, -t^{j-i} q^{\nu'_j - \nu_i} d/c; q)_\tau \end{aligned}$$

if  $(\nu, \nu') \in P_0(l; \varepsilon) \times P_1(m; \varepsilon)$ , and zero otherwise, with  $t = q^\tau$  and with

$$(3.8.11) \quad \begin{aligned} \Psi_0(\nu, l; a, b, c, d) &:= F_1(\nu) \\ &\cdot \prod_{i=1}^l \frac{1}{(q, -t^{1-i} q^{-\nu_{i-1}} d/c, at^{i-1} q^{\nu_{i-1}+1}, -bt^{i-1} q^{\nu_{i-1}+1} c/d; q)_\infty} \\ &\cdot \prod_{i=1}^l \frac{(at^{i-1} q^{\nu_{i-1}+1}, -bt^{i-1} q^{\nu_{i-1}+1} c/d; q)_{\nu_i - \nu_{i-1}}}{(q, -t^{i-1} q^{\nu_{i-1}+1} c/d; q)_{\nu_i - \nu_{i-1}} (abt^{i-1} q^{\nu_{i-1}+1})^{\nu_i - \nu_{i-1}}} \end{aligned}$$

where  $\nu_0 = 0$  and  $F_1(\nu)$  is given by (3.7.11), and

$$(3.8.12) \quad \begin{aligned} U_+(\varepsilon; \nu, \nu'; l, m) &:= \Psi_+(\varepsilon; \nu, l; a, b, c, d) \Psi_+(\varepsilon; \nu', m; b, a, d, c) \\ &\cdot \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} (-\varepsilon^2 t^{2-i-j} q^{-1-\nu_i - \nu'_j}; q)_\tau \end{aligned}$$



if  $(\nu, \nu') \in P_B^{(0)}(l; \varepsilon) \times P_B^{(1)}(m; \varepsilon)$  and zero otherwise, with

$$(3.8.13) \quad \Psi_+(\varepsilon; \nu, l; a, b, c, d) := \prod_{i=1}^l \frac{(\varepsilon^2 t^{2(1-i)} q^{-2\nu_{i-1}-1} d/c; q)_\infty}{(\varepsilon^2 a t^{1-i} q^{-\nu_{i-1}} d/c, -\varepsilon^2 b t^{1-i} q^{-\nu_{i-1}}; q)_\infty} \prod_{1 \leq i < j \leq l} (\varepsilon^2 t^{2-i-j} q^{-\nu_i - \nu_j - 1} d/c; q)_\tau$$

and

$$(3.8.14) \quad U_-(\varepsilon; \nu, \nu'; l, m) := \Psi_-(\varepsilon; \nu, l; a, b, c, d) \Psi_-(\varepsilon; \nu', m; b, a, d, c) \left( \prod_{j=1}^m \frac{(-\varepsilon^{-2} q t^{l+j-1}; q)_\infty}{(-\varepsilon^{-2} q t^{j-1}; q)_\infty} \right) \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} (-\varepsilon^{-2} t^{i+j-2} q^{\nu_i + \nu'_j + 1}; q)_\tau$$

if  $(\nu, \nu') \in P_B^{(0)}(l; \varepsilon) \times P_B^{(1)}(m; \varepsilon)$  and zero otherwise, with

$$(3.8.15) \quad \Psi_-(\varepsilon; \nu, l; a, b, c, d) := \prod_{i=1}^l \frac{(\varepsilon^{-2} t^{2(i-1)} q^{2\nu_i+1} c/d; q)_1}{(\varepsilon^{-2} t^{2(i-1)} q^{2\nu_{i-1}+1} c/d; q)_1} \prod_{i=1}^l \frac{(\varepsilon^{-2} t^{2(i-1)} q^{2\nu_{i-1}+1} c/d, -\varepsilon^{-2} t^{i-1} q^{\nu_{i-1}+1}; q)_{\nu_i - \nu_{i-1}}}{(\varepsilon^{-2} a^{-1} t^{i-1} q^{\nu_{i-1}+1} c/d, -\varepsilon^{-2} b^{-1} t^{i-1} q^{\nu_{i-1}+1}; q)_{\nu_i - \nu_{i-1}}} \frac{\prod_{i=1}^l (-\varepsilon^{-2} q t^{i-1}; q)_{\nu_{i-1}}}{\prod_{1 \leq i < j \leq l} (\varepsilon^{-2} t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1} c/d; q)_{\nu_i - \nu_{i-1}}}$$

For given  $\varepsilon_0 \in \mathbb{R}^*$ , we write  $\varepsilon_k := \varepsilon_0 q^k$ . Then for generic  $\varepsilon_0 > 0$  we have

$$(3.8.16) \quad \lim_{k \rightarrow \infty} U_+(\varepsilon_k; \nu, \nu'; l, m) = 1$$

for all  $(\nu, \nu') \in P(l) \times P(m)$  by Lemma 3.7.2(a). By (3.7.25), we have for generic  $\varepsilon_0 > 0$

$$(3.8.17) \quad \lim_{k \rightarrow \infty} \frac{\prod_{i=1}^l (-\varepsilon_k^{-2} q t^{i-1}; q)_{\nu_{i-1}}}{\prod_{1 \leq i < j \leq l} (\varepsilon_k^{-2} t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1} c/d; q)_{\nu_i - \nu_{i-1}}} = \prod_{i=1}^l q^{(\nu_{i-1} + 1)t^{(i-1)\nu_{i-1}}} \prod_{1 \leq i < j \leq l} (-t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1} c/d)^{\nu_{i-1} - \nu_i} q^{-(\nu_{i-1} - \nu_{i-1})}$$

for the factor of  $\Psi_-$  in the third line of (3.8.15). The factor of  $U_-$  in the second line of (3.8.14) can be rewritten as

$$(3.8.18) \quad \left( \prod_{j=1}^m \frac{(-\varepsilon^{-2}qt^{l+j-1}; q)_\infty}{(-\varepsilon^{-2}qt^{j-1}; q)_\infty} \right) \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} (-\varepsilon^{-2}t^{i+j-2}q^{\nu_i+\nu'_j+1}; q)_\tau \\ = \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} \frac{(-\varepsilon^{-2}qt^{i+j-1}; q)_{\nu_i+\nu'_j}}{(-\varepsilon^{-2}qt^{i+j-2}; q)_{\nu_i+\nu'_j}}.$$

It follows then from (3.8.14), (3.8.15), (3.8.17), (3.8.18) and Lemma 3.7.2(b) that for generic  $\varepsilon_0 > 0$ ,

$$(3.8.19) \quad \lim_{k \rightarrow \infty} U_-(\varepsilon_k; \nu, \nu'; l, m) = t^{m|\nu|+l|\nu'|} \Psi_-^\infty(\nu, l; a, b, c, d) \Psi_-^\infty(\nu', m; b, a, d, c)$$

with

$$(3.8.20) \quad \Psi_-^\infty(\nu, l; a, b, c, d) := \prod_{i=1}^l (t^{i-1}q^{\nu_{i-1}+2ab})^{\nu_i-\nu_{i-1}} q^{\binom{\nu_{i-1}+1}{2}} t^{(i-1)\nu_{i-1}} \\ \cdot \prod_{1 \leq i < j \leq l} (-t^{i+j-2}q^{\nu_{i-1}+\nu_j+1}c/d)^{\nu_{i-1}-\nu_i} q^{-\binom{\nu_i-\nu_{i-1}}{2}}.$$

We will rewrite now  $U_0$  in the form

$$(3.8.21) \quad U_0(\nu, \nu'; l, m) = (q; q)_\infty^{-r} \prod_{i=1}^l \frac{\theta(-t^{i-m}c/d)}{\theta(-t^{1-i}d/c)\theta(-t^i c/d)} \prod_{j=1}^m \frac{1}{\theta(-t^{1-j}c/d)} \\ \cdot C_0(\nu, \nu'; l, m) \Delta^B(\rho_B q^\nu, \sigma_B q^{\nu'}; a, b, c, d; t)$$

and we determine the factor  $C_0(\nu, \nu'; l, m)$  explicitly. Using (3.4.8) and using the formula  $\theta(x) = \theta(qx^{-1})$  for the Jacobi theta function  $\theta(x)$  (3.4.6),  $\Psi_0(\nu, l; a, b, c, d)$  (3.8.11) can be rewritten as

$$(3.8.22) \quad \Psi_0(\nu, l; a, b, c, d) = F_1(\nu) \\ \cdot \prod_{i=1}^l \frac{v_B(\rho_{B,i} q^{\nu_i}; a, b, c, d) (t^{i-1}c/d)^{\nu_{i-1}} q^{\binom{\nu_{i-1}+1}{2}}}{\theta(-t^{1-i}d/c) (q^{\nu_i+1}t^{i-1}; q)_\infty (q; q)_{\nu_i-\nu_{i-1}} (abt^{i-1}q^{\nu_{i-1}+1})^{\nu_i-\nu_{i-1}}}$$

where  $v_B$  (3.2.16) is the one variable weight function for the big  $q$ -Jacobi polynomials. Since  $v_B(-dx; a, b, c, d) = v_B(dx; b, a, d, c)$ , we obtain from (3.8.22)

$$(3.8.23) \quad \Psi_0(\nu', m; b, a, d, c) = F_1(\nu') \\ \cdot \prod_{j=1}^m \frac{v_B(\sigma_{B,j} q^{\nu'_j}; a, b, c, d) (t^{j-1}d/c)^{\nu'_{j-1}} q^{\binom{\nu'_{j-1}+1}{2}}}{\theta(-t^{1-j}c/d) (q^{\nu'_j+1}t^{j-1}; q)_\infty (q; q)_{\nu'_j-\nu'_{j-1}} (abt^{j-1}q^{\nu'_{j-1}+1})^{\nu'_j-\nu'_{j-1}}}.$$

The factor  $F_1(\nu)$  (3.7.11) of  $\Psi_0(\nu, l; a, b, c, d)$  can be rewritten as

$$(3.8.24) \quad F_1(\nu) = \delta_{qJ}(\rho_B q^\nu) q^{-2\tau^2 \binom{l}{3}} c^{-l(l-1)\tau} \prod_{i=1}^l \frac{(t^{i-1} q^{1+\nu_i}; q)_\infty}{(q^{1+\nu_i-\nu_{i-1}}; q)_\infty} t^{-2(l-i)\nu_i} \\ \cdot \prod_{1 \leq i < j \leq l} (-q^{\nu_j-\nu_i+1} t^{j-i})^{\nu_i-\nu_{i-1}} q^{\binom{\nu_i-\nu_{i-1}}{2}}$$

for  $\nu \in P(l)$  where  $\delta_{qJ}$  is given by (3.3.6) and  $t = q^\tau$ . This follows from (3.7.13) since  $\delta_{qJ}(\rho_B q^\nu) = c^{l(l-1)\tau} \delta_{qJ}(\rho_L q^\nu)$  for  $\nu \in P(l)$  (here  $\rho_{L,i} = t^{i-1}$ ). Similarly, we have for the factor  $F_1(\nu')$  of  $\Psi_0(\nu', m; b, a, d, c)$ ,

$$(3.8.25) \quad F_1(\nu') = \delta_{qJ}(\sigma_B q^{\nu'}) q^{-2\tau^2 \binom{m}{3}} d^{-m(m-1)\tau} \prod_{j=1}^m \frac{(t^{j-1} q^{1+\nu'_j}; q)_\infty}{(q^{1+\nu'_j-\nu'_{j-1}}; q)_\infty} t^{-2(m-j)\nu'_j} \\ \cdot \prod_{1 \leq i < j \leq m} (-q^{\nu'_j-\nu'_i+1} t^{j-i})^{\nu'_i-\nu'_{i-1}} q^{\binom{\nu'_i-\nu'_{i-1}}{2}},$$

since  $\delta_{qJ}(\sigma_B q^{\nu'}) = d^{m(m-1)\tau} \delta_{qJ}(\rho_L q^{\nu'})$  for  $\nu' \in P(m)$ .

Finally, we set for  $z = (z_1, \dots, z_r)$  with  $r := l + m$ ,

$$\tilde{\delta}_{qJ}(z) := \prod_{\substack{1 \leq i < l \\ l+1 \leq j \leq r}} |z_i - z_j| |z_i|^{2\tau-1} (qt^{-1} z_j / z_i; q)_{2\tau-1},$$

then the factor of  $U_0$  in the second line of (3.8.10) can be rewritten as

$$(3.8.26) \quad \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} (-t^{i-j} q^{\nu_i-\nu'_j} c/d, -t^{j-i} q^{\nu'_j-\nu_i} d/c; q)_\tau \\ = \tilde{\delta}_{qJ}(\rho_B q^\nu, \sigma_B q^{\nu'}) t^{m|\nu|-l|\nu'|} \prod_{i=1}^l \frac{\theta(-t^{i-m} c/d)}{\theta(-t^i c/d) (ct^{i-1} q^{\nu_i})^{2m\tau}}$$

for  $\nu \in P(l)$  and  $\nu' \in P(m)$ , since we have for  $i \in [1, l]$ ,  $j \in [1, m]$  that

$$(3.8.27) \quad (-t^{i-j} q^{\nu_i-\nu'_j} c/d, -t^{j-i} q^{\nu'_j-\nu_i} d/c; q)_\tau \\ = \frac{\theta(-t^{i-j} c/d) (-t^{j-i-1} q^{1+\nu'_j-\nu_i} d/c; q)_\infty}{\theta(-t^{i-j+1} c/d) (-t^{j-i+1} q^{\nu'_j-\nu_i} d/c; q)_\infty} (1 + t^{j-i} q^{\nu'_j-\nu_i} d/c) t^{\nu_i-\nu'_j}$$

(formula (3.8.27) follows from a straightforward computation using (2.5.3), (3.4.8) and  $\theta(x) = \theta(qx^{-1})$ ). Since

$$\delta_{qJ}(\rho_B q^\nu, \sigma_B q^{\nu'}) = \delta_{qJ}(\rho_B q^\nu) \delta_{qJ}(\sigma_B q^{\nu'}) \tilde{\delta}_{qJ}(\rho_B q^\nu, \sigma_B q^{\nu'})$$

for  $\nu \in P(l)$  and  $\nu' \in P(m)$ , we obtain from (3.4.3), (3.8.10), (3.8.22), (3.8.23), (3.8.24), (3.8.25) and (3.8.26) that (3.8.21) holds with

$$(3.8.28) \quad C_0(\nu, \nu'; l, m) = t^{-m|\nu|-l|\nu'|} c^{-2lm\tau} q^{-2m\binom{l}{2}\tau^2} \cdot \hat{C}_0(\nu, l; a, b, c, d) \hat{C}_0(\nu', m; b, a, d, c)$$

where

$$(3.8.29) \quad \begin{aligned} \hat{C}_0(\nu, l; a, b, c, d) &:= q^{-2\tau^2\binom{l}{3}} c^{-2\binom{l}{2}\tau} \\ &\cdot \prod_{i=1}^l (t^{i-1}c/d)^{\nu_{i-1}} q^{\binom{\nu_{i-1}+1}{2}} t^{-2(l-i)\nu_i} (abt^{i-1}q^{\nu_{i-1}+1})^{\nu_{i-1}-\nu_i} \\ &\cdot \prod_{1 \leq i < j \leq l} (-q^{\nu_j-\nu_i+1}t^{j-i})^{\nu_i-\nu_{i-1}} q^{\binom{\nu_i-\nu_{i-1}}{2}}. \end{aligned}$$

By Lemma 3.8.1 (with  $n$  in the right hand side of (3.8.1) equal to  $r = l + m$  in this situation), and by (3.8.9), (3.8.16), (3.8.19), (3.8.21) and (3.8.28) we have for generic  $\varepsilon_0 > 0$  that

$$(3.8.30) \quad \begin{aligned} \lim_{k \rightarrow \infty} \Delta_{1,l,m}^{KB}(\nu, \nu'; \varepsilon_k) &= U_0(\nu, \nu'; l, m) \lim_{k \rightarrow \infty} U_-(\varepsilon_k; \nu, \nu'; l, m) \\ &= (q; q)_{\infty}^{-2r} c_{B,l} \Delta^B(\rho_B q^{\nu}, \sigma_B q^{\nu'}; a, b, c, d; t) \\ &\quad \cdot E(\nu, l; a, b, c, d) E(\nu', m; b, a, d, c) \end{aligned}$$

for  $\nu \in P(l)$  and  $\nu' \in P(m)$ , with

$$(3.8.31) \quad E(\nu, l; a, b, c, d) := q^{2\tau^2\binom{l}{3}} t^{\binom{l}{2}} c^{2\binom{l}{2}\tau+l} \hat{C}_0(\nu, l; a, b, c, d) \Psi_{-}^{\infty}(\nu, l; a, b, c, d).$$

By (3.8.20) and (3.8.29), we obtain

$$E(\nu, l; a, b, c, d) = c^l t^{\binom{l}{2}} q^{|\nu|} = \prod_{i=1}^l \rho_{B,i} q^{\nu_i}, \quad \nu \in P(l).$$

In particular we have  $E(\nu', m; b, a, d, c) = \prod_{j=1}^m |\sigma_{B,j}| q^{\nu'_j}$  for  $\nu' \in P(m)$ , hence by (3.8.30)

$$\lim_{k \rightarrow \infty} \Delta_{1,l,m}^{KB}(\nu, \nu'; \varepsilon_k) = (q; q)_{\infty}^{-2r} c_{B,l} \Delta^B(\rho_B q^{\nu}, \sigma_B q^{\nu'}; a, b, c, d; t) \prod_{i=1}^l \rho_{B,i} q^{\nu_i} \prod_{j=1}^m |\sigma_{B,j}| q^{\nu'_j}$$

for all  $\nu \in P(l)$ ,  $\nu' \in P(m)$ . To complete the proof of Lemma 3.8.2(i), it suffices to prove that for generic  $\varepsilon_0 > \max((qc/d)^{\frac{1}{2}}, (qd/c)^{\frac{1}{2}})$ ,

$$\sup_{k \in \mathbb{Z}_+} |\Delta_{1,l,m}^{KB}(\nu, \nu'; \varepsilon_k)| \leq K c_{B,l} \Delta^B(\rho_B q^{\nu}, \sigma_B q^{\nu'}; a, b, c, d; t) \prod_{i=1}^l \rho_{B,i} q^{\nu_i} \prod_{j=1}^m |\sigma_{B,j}| q^{\nu'_j}$$

for all  $\nu \in P(l)$  and all  $\nu' \in P(m)$ , with  $K > 0$  independent of  $\nu$  and  $\nu'$ . This can be proved by similar arguments as in the little  $q$ -Jacobi case (see proof of Lemma 3.7.1(i)). In particular, the estimates for almost all factors of  $\Delta_{1,l,m}^{KB}$  can be obtained from one of

the three estimates of Lemma 3.7.2. Only for the factor in the third line of the expression of  $\Psi_-$  (3.8.15) one needs a separate argument to establish the desired estimate. We may assume that this factor is zero unless  $\nu \in P_B^{(0)}(l; \varepsilon)$  and  $\nu' \in P_B^{(1)}(m; \varepsilon)$ . In view of the limit (3.8.17), we would like to establish the estimate

$$\begin{aligned} & \sup_{\{k \in \mathbb{Z}_+ | \nu \in P_B^{(0)}(l; \varepsilon_k), \nu' \in P_B^{(1)}(m; \varepsilon_k)\}} \left| \frac{\prod_{i=1}^l (-\varepsilon_k^{-2} q t^{i-1}; q)_{\nu_{i-1}}}{\prod_{1 \leq i < j \leq l} (\varepsilon_k^{-2} t^{i+j-2} q^{\nu_j + \nu_{i-1} + 1} c/d; q)_{\nu_i - \nu_{i-1}}} \right| \\ & \leq K' \prod_{i=1}^l q^{\binom{\nu_{i-1} + 1}{2}} t^{(i-1)\nu_{i-1}} \prod_{1 \leq i < j \leq l} (t^{i+j-2} q^{\nu_{i-1} + \nu_j + 1} c/d)^{\nu_{i-1} - \nu_i} q^{-\binom{\nu_i - \nu_{i-1}}{2}} \end{aligned}$$

with  $K' > 0$  independent of  $\nu \in P(l)$  and  $\nu' \in P(m)$ , and  $\varepsilon_0 > \max((qc/d)^{\frac{1}{2}}, (qd/c)^{\frac{1}{2}})$  generic. This can be done similarly as was done for the factor  $N_1^3$  (3.7.19) in the little  $q$ -Jacobi case.

The proof of Lemma 3.8.2(ii) is similar to the proof of Lemma 3.7.1(iii).  $\square$



## CHAPTER 4

# Harmonic analysis on quantum Grassmannians

### 4.1. Introduction

In this chapter certain subfamilies of the Koornwinder polynomials and certain subfamilies of the multivariable big and little  $q$ -Jacobi polynomials are interpreted on quantum analogues of the complex Grassmannian

$$U/K := U(n)/(U(n-l) \times U(l)), \quad (l \leq [n/2]),$$

where  $U(n)$  is group of  $n$  by  $n$  unitary matrices.

The first connections between  $q$ -special functions and quantum groups were described in the papers [129], [87], [63], in which the little  $q$ -Jacobi polynomials were related to matrix coefficients of irreducible representations of the quantum analogue of  $SU(2)$ . In [101] Podles defined a one parameter family of quantum analogues of 2-spheres. The related zonal spherical functions were identified with big  $q$ -Jacobi polynomials by Noumi and Mimachi [93].

In [64] and [66], Koornwinder generalized these results for quantum 2-spheres by replacing the notion of invariance under a quantum subgroup by the notion of invariance under a twisted primitive element in the quantized universal enveloping algebra. The problem that the quantized function algebra on  $U$  has less quantum subgroups than one would expect from the classical setting is circumvented by this infinitesimal approach. In particular, the infinitesimal approach allowed Koornwinder to identify a two parameter family of Askey-Wilson polynomials as zonal spherical functions on the one-parameter family of Podles spheres. The special cases of big and little  $q$ -Jacobi polynomials could then be reobtained by sending one respectively two parameters to infinity. On the level of the spherical functions, this corresponds with the limit transitions from Askey-Wilson polynomials to big respectively little  $q$ -Jacobi polynomials as discussed in the previous chapter.

Analogous statements are valid for the projective space, see Noumi, Yamada and Mimachi [96] for the little  $q$ -Jacobi case, and Dijkhuizen and Noumi [24] for the general case. In this chapter, the generalizations of these results to the higher rank cases of the complex Grassmannian are considered.

Noumi [91] applied the infinitesimal method for the first time on higher rank symmetric spaces. The quantized symmetric spaces were now defined using invariance under certain 2-sided coideals in the quantized universal enveloping algebra. So far, the method has been successfully applied for all compact symmetric spaces of classical type, e.g. [94],

[95], [92], [124], [24]. In all cases, the zonal spherical functions can be identified with Koornwinder polynomials or Macdonald polynomials.

Noumi, Dijkhuizen and Sugitani [92] introduced a one parameter family of quantum analogues of the complex Grassmannian and they announced results on the corresponding harmonic analysis. The spherical functions associated with this family of quantum Grassmannians can be identified with a two parameter subfamily of the Koornwinder polynomials. The quantum subgroup case, which is defined using invariance with respect to the obvious quantum subgroup corresponding to  $K = U(n-l) \times U(l)$ , can be formally obtained from the one parameter family of quantum Grassmannians by sending the parameter to infinity. In this chapter it is shown that this type of limit transitions on quantum Grassmannians is compatible with the limit transitions from Koornwinder polynomials to multivariable big and little  $q$ -Jacobi polynomials, which were proved in the previous chapter. In particular, the zonal spherical functions on the quantum complex Grassmannian for the quantum subgroup case can be identified with multivariable big and little  $q$ -Jacobi polynomials. These results will follow from a careful study of the one parameter family of quantum Grassmannians which was introduced in [92] by Noumi, Dijkhuizen and Sugitani.

This chapter is organized as follows. In Section 4.2 the harmonic analysis on the classical complex Grassmannian is recalled. Section 4.3 contains preliminary results on the quantum unitary group and the corresponding quantized universal enveloping algebra. In Section 4.4 the spherical representations corresponding to the quantized complex Grassmannian (quantum subgroup case) are classified. In Section 4.5 and Section 4.6 we recall the announced results of Noumi, Dijkhuizen and Sugitani [92] about the harmonic analysis on a one parameter family of quantum Grassmannians, and we provide detailed proofs for most of these results. Finally, in Section 4.7 the limit transitions from Koornwinder polynomials to multivariable big and little  $q$ -Jacobi polynomials, cf. (3.3.13) respectively (3.4.15), are linked to limits of the one parameter family of quantum Grassmannians.

## 4.2. The classical complex Grassmannian

Two general references for the contents of this section are Helgason [41], and Heckman and Schlichtkrull [39].

Throughout this paper,  $n \geq 2$  and  $1 \leq l \leq \lfloor \frac{n}{2} \rfloor$  are fixed integers. Let  $G := GL(n, \mathbb{C})$  denote the general linear group with Lie algebra  $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ , and  $U := U(n)$  the unitary group with Lie algebra  $\mathfrak{u}$ . Let  $T \subset U$  denote the maximal torus consisting of diagonal matrices in  $U$ . Write  $\mathfrak{h} \subset \mathfrak{g}$  for the corresponding Cartan subalgebra. Let  $e_{ij}$  ( $1 \leq i, j \leq n$ ) denote the standard matrix units. The matrices  $h_i := e_{ii}$  ( $1 \leq i \leq n$ ) form a basis of  $\mathfrak{h}$ . Write  $\tilde{\varepsilon}_i \in \mathfrak{h}^*$  ( $1 \leq i \leq n$ ) for the corresponding dual basis vectors and define a non degenerate symmetric bilinear form on  $\mathfrak{h}^*$  by  $\langle \tilde{\varepsilon}_i, \tilde{\varepsilon}_j \rangle = \delta_{i,j}$ . The usual positive system  $R^+$  in the root system  $R := R(\mathfrak{g}, \mathfrak{h})$  consists of the vectors  $\tilde{\varepsilon}_i - \tilde{\varepsilon}_j$  ( $1 \leq i < j \leq n$ ). Let  $P = P_n := \bigoplus_{1 \leq i \leq n} \mathbb{Z}\tilde{\varepsilon}_i$  denote the rational character lattice of  $G$  (equivalently, the lattice of analytically integral weights of  $U$ ). Recall that the cone of



dominant weights  $P^+ = P_n^+$  is given by

$$(4.2.1) \quad P_n^+ := \{(\lambda_1, \dots, \lambda_n) \in P \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}.$$

Denote by  $\leq$  the (partial) dominance ordering on  $P$ . One has  $\mu \leq \lambda$  if and only if

$$(4.2.2) \quad \sum_{i=1}^j \mu_i \leq \sum_{i=1}^j \lambda_i \quad (1 \leq j \leq n-1) \quad \text{and} \quad \sum_{i=1}^n \mu_i = \sum_{i=1}^n \lambda_i.$$

We write  $K := U(n-l) \times U(l)$  and  $\mathfrak{k} := \mathfrak{gl}(n-l, \mathbb{C}) \oplus \mathfrak{gl}(l, \mathbb{C})$  for the corresponding complexified Lie algebra  $\mathfrak{k}$ .  $K$  is regarded as a subgroup of  $U$  via the embedding

$$(4.2.3) \quad U(n-l) \times U(l) \hookrightarrow U(n), \quad (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

The pair  $(U, K)$  is symmetric. Indeed, the involutive Lie group automorphism  $\theta : U \rightarrow U$ , defined by  $\theta(g) := JgJ$  with

$$(4.2.4) \quad J := \sum_{1 \leq k \leq n-l} e_{kk} - \sum_{1 \leq k \leq l} e_{k'k'}, \quad (k' := n+1-k)$$

has fixed point group  $K$ . The differential of  $\theta$  at the unit element  $e \in U$ , extended  $\mathbb{C}$ -linearly to a Lie-algebra involution  $\mathfrak{g} \rightarrow \mathfrak{g}$ , will also be denoted by  $\theta$ . The  $+1$ -eigenspace of the involution  $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$  is exactly the Lie-subalgebra  $\mathfrak{k}$ . Write  $\mathfrak{p}$  for the  $-1$ -eigenspace of  $\theta$ , then we have the eigenspace decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ .

For certain purposes, it is more convenient to consider the involution  $\theta' : \mathfrak{g} \rightarrow \mathfrak{g}$  defined by  $\theta'(X) := J'XJ'$  with

$$(4.2.5) \quad J' := \sum_{l < k < l'} e_{kk} - \sum_{1 \leq k \leq l} e_{kk'} - \sum_{1 \leq k \leq l} e_{k'k}.$$

Since  $J'$  is conjugate to  $J$ , the involution  $\theta'$  is conjugate to  $\theta$  by an inner automorphism of  $\mathfrak{g}$ . Observe that both  $\theta$  and  $\theta'$  leave  $\mathfrak{u} \subset \mathfrak{g}$  invariant.

Let  $\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{p}'$  be the eigenspace decomposition of  $\theta'$  in  $+1$  and  $-1$  eigenspaces. The intersection  $\mathfrak{h} \cap \mathfrak{k}'$  is spanned by the elements  $h_i + h_{i'}$  ( $1 \leq i \leq l$ ) and  $h_i$  ( $l < i < l'$ ), whereas the intersection  $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{p}'$  is spanned by  $h_i - h_{i'}$  ( $1 \leq i \leq l$ ) and is maximal abelian in  $\mathfrak{p}'$ . The positive system of  $R$  with respect to the lexicographic ordering of  $\mathfrak{h}_{\mathbb{R}} := \sum_{j=1}^n \mathbb{R}h_j$  relative to the ordered basis  $h_1 - h_{1'}, \dots, h_l - h_{l'}, h_1 + h_{1'}, \dots, h_l + h_{l'}, h_{l+1}, \dots, h_{n-l}$  coincides with  $R^+$ .

Write  $\varepsilon'_i$  for the restriction of  $\varepsilon_i$  to  $\mathfrak{a}$  ( $1 \leq i \leq l$ ). The root system  $R \subset \mathfrak{h}^*$  is mapped under the natural projection  $\mathfrak{h}^* \rightarrow \mathfrak{a}^*$  onto the restricted root system  $\Sigma' = \Sigma'(\mathfrak{g}, \mathfrak{a})$ . Choose the positive system in  $\Sigma'$  with respect to the lexicographic ordering of  $\mathfrak{a}_{\mathbb{R}} := \mathfrak{h}_{\mathbb{R}} \cap \mathfrak{a}$  relative to the ordered basis  $h_1 - h_{1'}, \dots, h_l - h_{l'}$  of  $\mathfrak{a}_{\mathbb{R}}$ . This ordering is compatible with the lexicographic ordering of  $\mathfrak{h}_{\mathbb{R}}$  introduced above in the sense that  $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$  is positive if its restriction to  $\mathfrak{a}_{\mathbb{R}}$  is strict positive. The positive root vectors in  $\Sigma'$  are

$$\varepsilon'_i \quad (1 \leq i \leq l), \quad \varepsilon'_i \pm \varepsilon'_j \quad (1 \leq i < j \leq l), \quad 2\varepsilon'_i \quad (1 \leq i \leq l),$$

the roots  $\varepsilon'_i$  ( $1 \leq i \leq l$ ) occurring only if  $n \neq 2l$ .  $\Sigma'$  is isomorphic with  $BC_l$  if  $n \neq 2l$  and isomorphic with  $C_l$  if  $n = 2l$ . The root multiplicities corresponding to the short, medium, and long roots are

$$(4.2.6) \quad m_1 = 2(n - 2l), \quad m_2 = 2 (l > 1), \quad m_3 = 1.$$

For later purposes, it is convenient to rescale the root system  $\Sigma'$  by a factor 2. So we set  $\Sigma := 2\Sigma' \subset \mathfrak{a}^*$ ,  $\varepsilon_i := 2\varepsilon'_i$  ( $1 \leq i \leq l$ ). Then the corresponding weight lattice  $P_\Sigma \subset \mathfrak{a}^*$  is the  $\mathbb{Z}$ -span of the  $\varepsilon_i$  ( $1 \leq i \leq l$ ), and the set  $P_\Sigma^+$  of dominant weights  $\mu = \sum_i \mu_i \varepsilon_i$  (taken with respect to the lexicographic ordering on  $\mathfrak{a}_\mathbb{R}$  introduced above) is characterized by the condition  $\mu_1 \geq \dots \geq \mu_l \geq 0$ . The dominance ordering  $\leq$  on  $P_\Sigma$  is explicitly given by

$$(4.2.7) \quad \mu \leq \lambda \iff \sum_{i=1}^j \mu_i \leq \sum_{i=1}^j \lambda_i \quad (1 \leq j \leq l).$$

Let  $K' \subset U$  denote the connected subgroup corresponding to  $\mathfrak{k}'$ . The symmetric pairs  $(U, K)$  and  $(U, K')$  are Gelfand pairs, i.e. every finite dimensional irreducible representation of  $U$  has at most one  $K$ -fixed vector up to scalar multiples. According to [41, Chapter V, Theorem 4.1], a highest weight  $\lambda \in P^+$  is  $K'$ -spherical, i.e. corresponds to a representation with a non-zero  $K'$ -fixed vector, if and only if the restriction of  $\lambda$  to  $\mathfrak{h} \cap \mathfrak{k}'$  is zero and the restriction of  $\lambda$  to  $\mathfrak{a}$  lies in  $P_\Sigma^+$ . Hence we get the following result.

**THEOREM 4.2.1.** *The set  $P_K^+ \subset P^+$  of  $K$ -spherical dominant weights consists of all dominant weights of the form*

$$\lambda := (\lambda_1, \dots, \lambda_l, 0, \dots, 0, -\lambda_l, \dots, -\lambda_1).$$

Write  $\lambda^\natural := (\lambda_1, \dots, \lambda_l)$  for a dominant weight  $\lambda \in P_K^+$ . The assignment  $\lambda \mapsto \lambda^\natural$  defines a bijection of  $P_K^+$  onto  $P_\Sigma^+$ . Let  $\varpi_r \in P_K^+$  be the spherical weight for which  $\varpi_r^\natural = (1^r)$ . Then  $P_K^+ = \bigoplus_{1 \leq r \leq l} \mathbb{Z}_+ \varpi_r$ . We will call  $\{\varpi_r\}_{r=1}^l$  the fundamental dominant spherical weights.

Let  $A$  denote the algebra of polynomial functions on  $U$ ,  $A(T)$  the algebra of polynomial functions on the maximal torus  $T$ .  $A(T)$  may be naturally identified with the algebra  $\mathbb{C}[z^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$  in  $n$  variables  $z_i$  ( $1 \leq i \leq n$ ) in the following way. Observe that  $T \simeq i\mathfrak{h}_\mathbb{R}/2\pi i\tilde{P}$  using the exponential mapping, where  $\tilde{P} := \bigoplus_{1 \leq j \leq n} \mathbb{Z} h_j$ . Then the coordinate functions  $z_j$  can be defined by  $z_j := e^{\tilde{\varepsilon}_j}$ , where for  $[X] \in T$  with  $X \in i\mathfrak{h}_\mathbb{R}$  a representative of  $[X]$ ,  $e^{\tilde{\varepsilon}_j}([X]) := e^{\tilde{\varepsilon}_j(X)}$ . More explicitly,  $z_j$  is given by

$$z_j : \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \mapsto e^{i\theta_j}, \quad (\theta_k \in [0, 2\pi))$$

where  $\text{diag}(a_1, \dots, a_n)$  is the diagonal matrix with  $a_1, \dots, a_n$  on the diagonal.

Let  $\mathcal{H} \subset A$  denote the subalgebra of  $K'$ -biinvariant functions. One has the decomposition

$$(4.2.8) \quad \mathcal{H} = \bigoplus_{\lambda \in P_K^+} \mathcal{H}(\lambda), \quad \mathcal{H}(\lambda) := \mathcal{H} \cap W(\lambda),$$

$W(\lambda) \subset A$  denoting the subspace spanned by the matrix coefficients of the irreducible representation of highest weight  $\lambda$ . Each of the subspaces  $\mathcal{H}(\lambda)$  ( $\lambda \in P_K^+$ ) is one dimensional, since  $(U, K')$  is a Gelfand pair. Any non-zero element  $\varphi(\lambda)$  of  $\mathcal{H}(\lambda)$  is called a zonal spherical function.

Set

$$(4.2.9) \quad T_l := \exp(i\mathfrak{a}_{\mathbb{R}}) / (\exp(i\mathfrak{a}_{\mathbb{R}}) \cap K') \simeq i\mathfrak{a}_{\mathbb{R}} / 2\pi i Q_{\Sigma}^{\vee},$$

with  $Q_{\Sigma}^{\vee} \subset \mathfrak{a}_{\mathbb{R}}$  the coroot lattice of  $\Sigma$ . More explicitly, the coroot lattice  $Q_{\Sigma}^{\vee}$  is the  $\mathbb{Z}$ -span of the elements  $\frac{1}{2}(h_i - h_{i'})$  ( $i \in [1, l]$ ),  $\exp(i\mathfrak{a}_{\mathbb{R}})$  are the diagonal matrices

$$\text{diag}(e^{i\theta_1}, \dots, e^{i\theta_l}, 1, \dots, 1, e^{-i\theta_1}, \dots, e^{-i\theta_l}) \quad (\theta_j \in [0, 2\pi)),$$

and  $\exp(i\mathfrak{a}_{\mathbb{R}}) \cap K'$  are the matrices in  $\exp(i\mathfrak{a}_{\mathbb{R}})$  of order 2.

Write  $\log : T_l \rightarrow i\mathfrak{a}_{\mathbb{R}}$  for the multi-valued inverse of the exponential map  $\exp : i\mathfrak{a}_{\mathbb{R}} \rightarrow T_l$ . Similarly as for  $A(T)$ , the algebra of polynomial functions on  $T_l$  may be identified with the algebra  $\mathbb{C}[x^{\pm 1}]$  of Laurent polynomials in the  $l$  variables  $x_j$  ( $1 \leq j \leq l$ ), where the identification is given by  $x_j(t) := e^{\varepsilon_j(\log(t))}$ . In other words, the map  $x_j$  is given by

$$x_j : \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_l}, 1, \dots, 1, e^{-i\theta_1}, \dots, e^{-i\theta_l}) \mapsto e^{2i\theta_j}.$$

It follows that the algebra  $\mathbb{C}[x^{\pm 1}]$  of polynomial functions on  $T_l$  can be naturally embedded in the algebra  $\mathbb{C}[z^{\pm 1}]$  of polynomial functions on the maximal torus  $T$  by the assignment

$$(4.2.10) \quad x_1 = z_1 z_n^{-1}, \quad x_2 = z_2 z_{n-1}^{-1}, \quad \dots, \quad x_l = z_l z_{n+1-l}^{-1}.$$

Let  $\mathfrak{S} := \mathfrak{S}_l$  denote the permutation group on  $l$  letters,  $\mathcal{W} := \mathcal{W}_l = \mathbb{Z}_2^l \rtimes \mathfrak{S}_l$  the Weyl group of  $\Sigma$ . The natural action of  $\mathcal{W}$  on  $\mathfrak{a}_{\mathbb{R}}$  descends to  $T_l$ . Hence  $\mathcal{W}$  acts naturally on the algebra  $\mathbb{C}[x^{\pm 1}]$ . Write  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$  for the subalgebra of  $\mathcal{W}$ -invariant Laurent polynomials. By Chevalley's restriction theorem and the above mentioned natural embedding of  $\mathbb{C}[x^{\pm 1}]$  into  $\mathbb{C}[z^{\pm 1}]$ , we have the following theorem.

**THEOREM 4.2.2.** *Restriction to  $T$  induces an isomorphism of  $\mathcal{H}$  onto the algebra  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$  of  $\mathcal{W}$ -invariant Laurent polynomials in the variables  $x_i$  ( $1 \leq i \leq l$ ).*

Recall from the previous chapter that the  $BC$  type Heckman-Opdam polynomials  $\{P_{\lambda}^{HO}(\cdot; k)\}_{\lambda \in P_{\Sigma}^+}$  for  $k = (k_1, k_2, k_3) \in V_{HO}$  form a linear basis of  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$ , and that they are orthogonal with respect to the inner product

$$(4.2.11) \quad \langle P, Q \rangle_{HO} := \int_{T_l} P(t) \overline{Q(t)} \Delta_{HO}(t; k) dt.$$

Here  $dt$  denotes the normalized Haar measure on the torus  $T_l$  and  $t \mapsto \Delta_{HO}(t; k)$  on  $T_l$  is defined by

$$(4.2.12) \quad \Delta_{HO}(t; k) := \prod_{\alpha \in \Sigma} \left( e^{\frac{1}{2}\langle \alpha, \log(t) \rangle} - e^{-\frac{1}{2}\langle \alpha, \log(t) \rangle} \right)^{k_{\alpha}}.$$

The multiplicity parameters  $k_\alpha$  are by definition equal to  $k_i$  ( $i = 1, 2, 3$ ), depending on whether  $\alpha \in \Sigma$  is a short, medium, or long root. If  $l = 1$  there is no dependence on  $k_2$ .

In the following theorem the zonal spherical functions on the symmetric space  $U/K$  are identified with  $BC$  type Heckman-Opdam polynomials (cf. [39, p. 76]).

**THEOREM 4.2.3.** *Under restriction to the maximal torus  $T$ , the zonal  $K'$ -spherical function  $\varphi(\lambda)$  ( $\lambda \in \dot{P}_K^+$ ) is mapped onto (a scalar multiple of) the Heckman-Opdam hypergeometric polynomial  $P_{\lambda^{\mathfrak{h}}}^{HO}(x; k)$  with  $k_i = \frac{1}{2}m_i$  ( $i = 1, 2, 3$ ).*

Of course, the zonal  $K$ -spherical functions can be described in the same way, since the subgroups  $K$  and  $K'$  are conjugate.

As was shown in the previous chapter, the  $BC$  type Heckman-Opdam polynomials and the generalized Jacobi polynomials  $\{P_\lambda^J(\cdot; \alpha, \beta; \tau) \mid \lambda \in P_\Sigma^+\}$  are related via a simple change of variables,

$$(4.2.13) \quad P_\lambda^{HO}(x; k) = (-4)^{|\lambda|} P_\lambda^J(z(x); k_1 + k_3 - 1/2, k_3 - 1/2, k_2)$$

where  $z_j(x) := -\frac{1}{4}(x_j + x_j^{-1} - 2)$ . In particular, the zonal spherical functions in Theorem 4.2.3 under this change of variables become generalized Jacobi polynomials with parameter values  $\alpha = n - 2l$ ,  $\beta = 0$  and  $\tau = 1$ .

Being matrix coefficients of irreducible representations, the zonal spherical functions are mutually orthogonal with respect to the  $L^2$  inner product on  $A \subset L^2(U, dg)$ , where  $dg$  is the normalized Haar measure on  $U$  (Schur orthogonality). The restriction of the  $L^2$  inner product to the algebra  $\mathcal{H}$  of bi- $K'$ -invariant matrix coefficients coincides under the isomorphism of Theorem 4.2.2 with the inner product  $\langle \cdot, \cdot \rangle_{HO}$  on  $\mathbb{C}[x^\pm]^\mathcal{W}$  up to a non-zero positive constant. This non-zero positive constant can be explicitly determined using the well-known evaluation of the Selberg integral (cf. [107], [84]).

### 4.3. Preliminaries on the quantum unitary group

Various aspects of the quantum unitary group have been studied in many different papers. Our main references will be [96] and [91, Section 1], which are based on the  $R$ -matrix approach described in [104].

The quantized coordinate ring  $A_q(\text{Mat}(n, \mathbb{C}))$  of the space of  $n \times n$  complex matrices is defined as the algebra with generators  $t_{ij}$  ( $1 \leq i, j \leq n$ ) and relations

$$(4.3.1) \quad \begin{aligned} t_{ki}t_{kj} &= qt_{kj}t_{ki}, & t_{ik}t_{jk} &= qt_{jk}t_{ik} \quad (i < j), \\ t_{il}t_{kj} &= t_{kj}t_{il}, & t_{ij}t_{kl} - t_{kl}t_{ij} &= (q - q^{-1})t_{il}t_{kj} \quad (i < k, j < l). \end{aligned}$$

These equations can be written in more compact notation using the matrix

$$(4.3.2) \quad R := \sum_{ij} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}$$

where  $e_{ij}$  ( $1 \leq i, j \leq n$ ) denote the standard matrix units. Let  $V$  denote the vector space  $\mathbb{C}^n$  with canonical basis  $\{v_i\}_i$  and regard  $e_{ij}$  as matrix units in  $\text{End}(V)$  with respect to

the canonical basis  $\{v_i\}_i$ . The relations (4.3.1) are then given by the matrix equation  $(R \otimes 1)T_1T_2 = T_2T_1(R \otimes 1)$ , where

$$T_1 := \sum_{i,j} e_{ij} \otimes \text{id}_V \otimes t_{ij}, \quad T_2 := \sum_{i,j} \text{id}_V \otimes e_{ij} \otimes t_{ij}.$$

We will use the shorthand notation  $RT_1T_2 = T_2T_1R$  for this matrix equation. Observe furthermore that the matrix  $R$  is invertible, with inverse given by

$$R^{-1} := \sum_{ij} q^{-\delta_{ij}} e_{ii} \otimes e_{jj} - (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji}$$

and that  $R$  is a solution of the Quantum Yang-Baxter Equation

$$(4.3.3) \quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where  $R_{12} = R \otimes \text{id}_V$ ,  $R_{23} = \text{id}_V \otimes R$  and

$$R_{13} = \sum_{i,i'} r_i \otimes \text{id}_V \otimes r_{i'},$$

if  $R = \sum_i r_i \otimes r_i'$ .

A linear basis for  $A_q(\text{Mat}(n, \mathbb{C}))$  (cf. [96, Theorem 1.4], see also [60]) is given by the monomials

$$(4.3.4) \quad t^A := \prod_{(i,j)} t_{ij}^{a_{ij}} = t_{11}^{a_{11}} \cdots t_{1n}^{a_{1n}} t_{21}^{a_{21}} \cdots t_{nn}^{a_{nn}},$$

where  $A = (a_{ij})$  runs through the set  $\text{Mat}(n, \mathbb{Z}_+)$  of  $n \times n$  matrices with non-negative integer coefficients. We call  $\{t_A \mid A \in \text{Mat}(n, \mathbb{Z}_+)\}$  the monomial basis of  $A_q(\text{Mat}(n, \mathbb{C}))$ . A total order  $\preceq$  on  $\text{Mat}(n, \mathbb{Z}_+)$  is defined by associating with  $A \in \text{Mat}(n, \mathbb{Z}_+)$  the sequence

$$(4.3.5) \quad \left( \sum_{i,j} a_{ij}, a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{nn} \right) \in \mathbb{Z}_+^{n^2+1},$$

and ordering these sequences lexicographically. Every  $0 \neq \phi \in A_q(\text{Mat}(n, \mathbb{C}))$  can be written uniquely as

$$(4.3.6) \quad \phi = c_\phi t^{d(\phi)} + \text{lower order terms w.r.t. } \preceq$$

with  $d(\phi) \in \text{Mat}(n, \mathbb{Z}_+)$  and  $c_\phi \neq 0$ . We call  $c_\phi t^{d(\phi)}$  the leading term of  $\phi$ , and  $d(\phi)$  its degree. One easily sees that

$$(4.3.7) \quad d(\phi\psi) = d(\phi) + d(\psi), \quad 0 \neq \phi, \psi \in A_q(\text{Mat}(n, \mathbb{C}))$$

(cf. [96, Lemma 1.5]). In particular, there are no zero divisors in  $A_q(\text{Mat}(n, \mathbb{C}))$ .

The quantized coordinate ring  $A_q(G)$  of the general linear group  $G = GL(n, \mathbb{C})$  is defined by adjoining to  $A_q(\text{Mat}(n, \mathbb{C}))$  the inverse  $\det_q^{-1}$  of the quantum determinant

$$\det_q := \sum_{\sigma \in \mathfrak{S}_n} (-q)^{l(\sigma)} t_{1\sigma(1)} \cdots t_{n\sigma(n)} \in A_q(\text{Mat}_q(n, \mathbb{C}))$$

( $l$  denoting the length function on  $\mathfrak{S}$ ), which is central in  $A_q(\text{Mat}(n, \mathbb{C}))$ . The monomial basis of  $A_q(\text{Mat}(n, \mathbb{C}))$  can be naturally extended to a basis of  $A_q(G)$ , which we shall also call the monomial basis of  $A_q(G)$ .

There is a unique Hopf algebra structure on  $A_q(G)$  such that  $(t_{ij})$  becomes a matrix corepresentation, i.e.

$$\Delta(t_{ij}) = \sum_{k=1}^n t_{ik} \otimes t_{kj}, \quad \varepsilon(t_{ij}) = \delta_{ij}.$$

The antipode  $S: A_q(G) \rightarrow A_q(G)$  is given on the generators by

$$S(t_{ij}) := (-q)^{i-j} \xi_{i^c}^{j^c} \det_q^{-1}$$

with  $i^c := [1, n] \setminus \{i\}$ , and with the quantum minor  $\xi_J^I$  for subsets  $I = \{i_1 < \dots < i_r\}$ ,  $J = \{j_1 < \dots < j_r\} \subset [1, n]$  defined by

$$\xi_J^I := \sum_{\sigma \in \mathfrak{S}_r} (-q)^{l(\sigma)} t_{i_1 j_{\sigma(1)}} \cdots t_{i_r j_{\sigma(r)}}.$$

It is easy to see that the  $\xi_J^I$  are all non-zero.  $A_q(G)$  becomes a Hopf  $*$ -algebra by requiring  $(t_{ij})$  to be a unitary matrix corepresentation, i.e.  $t_{ij}^* := S(t_{ji})$ . We write  $A_q(U) = A_q(U(n))$  for  $A_q(G)$  endowed with this  $*$ -operation. The mapping  $\tau := * \circ S$  is a conjugate linear involution on  $A_q(U)$  such that  $\tau(t_{ij}) = t_{ji}$ .

The quantized Borel subgroups  $A_q(B^\pm)$  of upper respectively lower triangular matrices are defined as the Hopf quotients of  $A_q(G)$  by the relations

$$t_{ij} = 0 \quad (i > j) \quad \text{respectively} \quad t_{ij} = 0 \quad (i < j).$$

The corresponding projections will be denoted by  $\pi_\pm: A_q(G) \rightarrow A_q(B^\pm)$ . Observe that the  $z_i := \pi_\pm(t_{ii})$  ( $1 \leq i \leq n$ ) in  $A_q(B^\pm)$  are invertible. Corresponding to the diagonal subgroup  $T \subset U(n)$  we have a natural surjective Hopf  $*$ -algebra morphism  $|_T$  of  $A_q(U)$  onto the Laurent polynomial algebra  $A(T) := \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ .

Next, we briefly recall the global description of finite dimensional irreducible corepresentations of  $A_q(U)$ . For every  $\lambda \in P$ ,  $z^\lambda := z_1^{\lambda_1} \cdots z_n^{\lambda_n}$  defines a linear character (i.e. one dimensional corepresentation) of  $A_q(B^\pm)$  and of  $A(T)$ . Using these linear characters it is straightforward to define (highest) weight vectors in left or right  $A_q(U)$ -comodules. We take highest weight vectors of right and left  $A_q(U)$ -comodules with respect to  $A_q(B^+)$  and  $A_q(B^-)$ , respectively. For instance, a highest weight vector of weight  $\lambda \in P^+$  in a right  $A_q(U)$ -comodule  $M$  is a non-zero vector  $v \in M$  such that

$$(\text{id} \otimes \pi_+) \circ \rho_M(v) = v \otimes z^\lambda,$$

$\rho_M: M \rightarrow M \otimes A_q(U)$  denoting the comodule mapping. Finite dimensional  $A_q(U)$ -comodules are then completely reducible and unitarizable (see, for instance, [96], [23]). Recall that a right  $A_q(U)$ -comodule  $M$  endowed with a positive definite inner product

(taken to be conjugate linear in the second variable) is called unitary if

$$\sum_{(v)(w)} \langle v_{(1)}, w_{(1)} \rangle w_{(2)}^* v_{(2)} = \langle v, w \rangle 1 \quad (v, w \in M),$$

where the symbolic notation  $\sum_{(v)} v_{(1)} \otimes v_{(2)} := \rho_M(v)$  is used. The irreducible finite dimensional  $A_q(U)$ -comodules are parametrized by dominant weights  $\lambda \in P^+$  as in the classical case (cf. [96, Theorem 2.12]). The irreducible right  $A_q(U)$ -comodule with highest weight  $\lambda \in P^+$  is denoted by  $V_R(\lambda)$ . The vector space  $V_L(\lambda) := \text{Hom}(V_R(\lambda), \mathbb{C})$  has a natural left  $A_q(U)$ -comodule structure, which is also irreducible of highest weight  $\lambda$ . If no confusion is possible, we will write  $V(\lambda)$  for the left comodule  $V_L(\lambda)$  respectively for the right comodule  $V_R(\lambda)$ .

REMARK 4.3.1. Let  $M$  be a finite dimensional right  $A_q(U)$ -comodule with comodule mapping  $R: M \rightarrow M \otimes A_q(U)$ . Write  $M^\circ$  for the vector space complex conjugate to  $M$ ,  $\sigma: M \otimes A_q(U) \rightarrow A_q(U) \otimes M$  for the flip. Then the mapping

$$(4.3.8) \quad R^\circ: M^\circ \rightarrow A_q(U) \otimes M^\circ, \quad R^\circ := (\tau \otimes \text{id}) \circ \sigma \circ R,$$

where  $\tau = * \circ S$ , defines a left  $A_q(U)$ -comodule structure on  $M^\circ$ . In (4.3.8)  $R$  is considered as a conjugate linear map from  $M^\circ$  to  $M \otimes A_q(U)$  and  $\tau \otimes \text{id}$  as a conjugate linear map from  $A_q(U) \otimes M$  to  $A_q(U) \otimes M^\circ$ .

The assignment  $M \mapsto M^\circ$  is a 1-1 correspondence between right and left  $A_q(U)$ -comodules preserving weights and highest weights. Hence  $M^\circ$  is isomorphic to the left  $A_q(U)$ -comodule  $\text{Hom}(M, \mathbb{C})$ . A right  $A_q(U)$ -comodule intertwiner  $\Psi: M \rightarrow N$  also intertwines the left  $A_q(U)$ -comodule structures of  $M^\circ$  and  $N^\circ$  (i.e. when  $\Psi$  is considered as map from  $M^\circ$  to  $N^\circ$ ).

Recall that the comultiplication  $\Delta: A_q(U) \rightarrow A_q(U) \otimes A_q(U)$  defines a bicomodule structure on  $A_q(U)$ . Let  $W(\lambda) \subset A_q(U)$  ( $\lambda \in P^+$ ) denote the subspace spanned by the matrix coefficients of either  $V_R(\lambda)$  or  $V_L(\lambda)$ . The irreducible decomposition of the bicomodule  $A_q(U)$  reads

$$(4.3.9) \quad A_q(U) = \bigoplus_{\lambda \in P^+} W(\lambda), \quad W(\lambda) \simeq V_L(\lambda) \otimes V_R(\lambda).$$

Let  $h$  be the normalized Haar functional on  $A_q(U)$ . It can be characterized as the unique linear functional on  $A_q(U)$  which is zero on  $W(\lambda)$  for  $0 \neq \lambda \in P^+$  and which sends  $1 \in A_q(U)$  to  $1 \in \mathbb{C}$ . The subspaces  $W(\lambda)$  are mutually orthogonal with respect to the inner product  $\langle \varphi, \psi \rangle := h(\psi^* \varphi)$ .

We consider now in some more detail the vector corepresentation, its dual representation, and their exterior powers. Recall the notation  $\{v_i\}_i$  for the canonical basis of  $V = \mathbb{C}^n$ .  $V$  becomes a right  $A_q(U)$ -comodule (called vector corepresentation) with

$$(4.3.10) \quad R: V \mapsto V \otimes A_q(U), \quad R(v_j) := \sum_{i=1}^n v_i \otimes t_{ij} \quad (1 \leq j \leq n).$$

$V$  is irreducible with highest weight  $\tilde{\varepsilon}_1$  and highest weight vector  $v_1$ . Observe that the vectors  $v_i$  have weight  $\tilde{\varepsilon}_i$ . The corepresentation  $V$  is unitary with respect to the inner product  $\langle v_i, v_j \rangle = \delta_{ij}$ .

Let  $V^*$  denote the linear dual of  $V$  with dual basis  $(v_i^*)$ .  $V^*$  becomes a right  $A_q(U)$ -comodule (the dual corepresentation) with

$$(4.3.11) \quad R: V^* \mapsto V^* \otimes A_q(U), \quad R(v_j^*) := \sum_{i=1}^n v_i^* \otimes t_{ij}^* \quad (1 \leq j \leq n).$$

$V^*$  is irreducible with highest weight  $-\tilde{\varepsilon}_n$  and highest weight vector  $v_n^*$ . Note that the vectors  $v_i^*$  have weight  $-\tilde{\varepsilon}_i$ . The corepresentation  $V^*$  is unitary with respect to the inner product  $\langle v_i^*, v_j^* \rangle := q^{-(2\rho, \tilde{\varepsilon}_i)} \delta_{ij}$ , where  $\rho := \sum_{k=1}^n (n-k)\tilde{\varepsilon}_k$ . This follows from the well-known fact that  $S^2(t_{ij}) = q^{(2\rho, \tilde{\varepsilon}_j - \tilde{\varepsilon}_i)} t_{ij}$  ( $1 \leq i, j \leq n$ ).

Let  $\Lambda_q(V)$  respectively  $\Lambda_q(V^*)$  denote the associative algebra which is generated by  $v_1, \dots, v_n$  respectively  $v_1^*, \dots, v_n^*$  with relations

$$(4.3.12) \quad v_i \wedge v_i = 0 \quad (1 \leq i \leq n), \quad v_i \wedge v_j = -q^{-1} v_j \wedge v_i \quad (i < j)$$

respectively

$$(4.3.13) \quad v_i^* \wedge v_i^* = 0 \quad (1 \leq i \leq n), \quad v_i^* \wedge v_j^* = -q^{-1} v_j^* \wedge v_i^* \quad (i < j).$$

Then  $\Lambda_q(V)$  respectively  $\Lambda_q(V^*)$  inherits a natural right  $A_q(U)$ -comodule structure from  $V$  respectively  $V^*$  by extending the comodule mapping  $R$  as a unital algebra homomorphism.  $\Lambda_q(V)$  respectively  $\Lambda_q(V^*)$  has a natural grading such that the generators  $v_i$  respectively  $v_i^*$  have degree 1:

$$\Lambda_q(V) = \bigoplus_{r=0}^n \Lambda_q^r(V), \quad \Lambda_q(V^*) = \bigoplus_{r=0}^n \Lambda_q^r(V^*).$$

Write  $v_I := v_{i_1} \wedge \dots \wedge v_{i_r}$  and  $v_I^* := v_{i_r}^* \wedge \dots \wedge v_{i_1}^*$  if  $I = \{i_1 < \dots < i_r\} \subset [1, n]$ . Write  $|I|$  for the cardinality of  $I$ . Then the  $v_I$  and  $v_I^*$  ( $|I| = r$ ) form a basis of  $\Lambda_q^r(V)$  and  $\Lambda_q^r(V^*)$  respectively. One has the multiplicative property ( $I, J \subset [1, n]$ )

$$v_I \wedge v_J = s_q(I; J) v_{I \cup J}, \quad v_I^* \wedge v_J^* := s_q(J; I) v_{I \cup J}^*$$

where

$$s_q(I; J) := \begin{cases} 0 & \text{if } I \cap J \neq \emptyset, \\ (-q)^{l(I; J)} & \text{if } I \cap J = \emptyset, \end{cases}$$

and  $l(I; J) := |\{(i, j) \in I \times J \mid i > j\}|$ . The comodules  $\Lambda_q^r(V)$  and  $\Lambda_q^r(V^*)$  are irreducible subcomodules of  $\Lambda_q(V)$ , and the coactions satisfy

$$(4.3.14) \quad R(v_J) = \sum_{|I|=r} v_I \otimes \xi_J^I, \quad R(v_J^*) = \sum_{|I|=r} v_I^* \otimes (\xi_J^I)^* \quad (|J| = r).$$



From this it follows immediately that

$$(4.3.15) \quad \Delta(\xi_J^I) = \sum_{|K|=r} \xi_K^I \otimes \xi_J^K \quad (|I|, |J| = r)$$

and hence

$$(4.3.16) \quad \sum_{|K|=r} \xi_K^I S(\xi_J^K) = \delta_{I,J} \quad (|I|, |J| = r).$$

The  $A_q(U)$ -comodule  $\Lambda_q^r(V)$  respectively  $\Lambda_q^r(V^*)$  has highest weight  $(1^r) := \tilde{\varepsilon}_1 + \cdots + \tilde{\varepsilon}_r$  respectively  $-\tilde{\varepsilon}_{n-r+1} - \cdots - \tilde{\varepsilon}_n$  with highest weight vector  $v_{[1,r]} = v_1 \wedge \cdots \wedge v_r$  respectively  $v_{[n-r+1,n]}^* = v_n^* \wedge \cdots \wedge v_{n-r+1}^*$ . The inner product on  $\Lambda_q^r(V)$  defined by  $\langle v_I, v_J \rangle = \delta_{I,J}$  is  $A_q(U)$ -invariant. On the space  $\Lambda_q^r(V^*)$  we have the invariant inner product  $\langle v_I^*, v_J^* \rangle := \delta_{I,J} q^{-(2\rho, \tilde{\varepsilon}_I)}$ , where  $\tilde{\varepsilon}_I := \sum_{i \in I} \tilde{\varepsilon}_i$ .

The Laplace expansions for quantum minor determinants now easily follow from the fact that  $\Lambda_q(V)$  is an  $A_q(U)$ -comodule algebra (cf. [96, Proposition 1.1]). Fix  $I, J \subset [1, n]$  such that  $|I| = |J| > 0$ . Fix two subsets  $J_1, J_2 \subset J$  such that  $|J_1| + |J_2| = |J|$ . Then

$$(4.3.17) \quad s_q(J_1; J_2) \xi_J^I = \sum_{\substack{I_1, I_2 \subset I; \\ |I_1|=|J_1|, |I_2|=|J_2|}} s_q(I_1; I_2) \xi_{J_1}^{I_1} \xi_{J_2}^{I_2},$$

$$(4.3.18) \quad s_q(J_1; J_2) \xi_I^J = \sum_{\substack{I_1, I_2 \subset I; \\ |I_1|=|J_1|, |I_2|=|J_2|}} s_q(I_1; I_2) \xi_{I_1}^{J_1} \xi_{I_2}^{J_2}.$$

It can now be shown that (cf. [96, (3.2)])

$$(4.3.19) \quad (\xi_J^I)^* = S(\xi_I^J) = \frac{s_q(J; J^c)}{s_q(I; I^c)} \xi_{J^c}^{I^c} \det_q^{-1} \quad (|I| = |J| = r),$$

where  $I^c := \{1, \dots, n\} \setminus I$ .

For the following results on the quantized universal enveloping algebra associated with  $\mathfrak{gl}(n, \mathbb{C})$  we refer the reader to Noumi's paper [91, Section 1] and references therein. Let  $U_q(\mathfrak{g}) = U_q(\mathfrak{gl}(n, \mathbb{C}))$  denote the quantized universal enveloping algebra (cf. Drinfeld [27], Jimbo [47]) associated with the Lie algebra  $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ . The algebra  $U_q(\mathfrak{g})$  is generated by elements  $q^h$  ( $h \in P$ ) and  $X_i^\pm$  ( $1 \leq i \leq n-1$ ) subject to the relations

$$(4.3.20) \quad \begin{aligned} q^0 &= 1, & q^h q^{h'} &= q^{h+h'} \\ q^h X_i^\pm q^{-h} &= q^{\pm(h, \alpha_i)} X_i^\pm \\ [X_i^+, X_j^-] &= \delta_{i,j} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}} \\ (X_i^\pm)^2 X_j^\pm - (q + q^{-1}) X_i^\pm X_j^\pm X_i^\pm + X_j^\pm (X_i^\pm)^2 &= 0, \quad (|i - j| = 1) \\ X_i^\pm X_j^\pm &= X_j^\pm X_i^\pm, \quad (|i - j| > 1), \end{aligned}$$

where  $\alpha_i := \tilde{\varepsilon}_i - \tilde{\varepsilon}_{i+1}$  ( $i \in [1, n-1]$ ) are the simple roots for  $R = R(\mathfrak{g}, \mathfrak{h})$ . The following formulas uniquely determine a Hopf- $*$ -algebra structure on  $U_q(\mathfrak{g})$ :

$$(4.3.21) \quad \begin{aligned} \Delta(X_i^+) &= X_i^+ \otimes 1 + q^{\alpha_i} \otimes X_i^+, \quad \Delta(X_i^-) = X_i^- \otimes q^{-\alpha_i} + 1 \otimes X_i^-, \\ \Delta(q^h) &= q^h \otimes q^h, \\ S(q^h) &= q^{-h}, \quad S(X_i^+) = -q^{-\alpha_i} X_i^+, \quad S(X_i^-) = -X_i^- q^{\alpha_i}, \\ \varepsilon(q^h) &= 1, \quad \varepsilon(X_i^\pm) = 0, \\ (q^h)^* &= q^h, \quad (X_i^+)^* = q^{-1} X_i^- q^{\alpha_i}, \quad (X_i^-)^* = q q^{-\alpha_i} X_i^{-1}. \end{aligned}$$

More useful for the purposes of this chapter are the  $L$ -operators  $L_{ij}^\pm \in U_q(\mathfrak{g})$  ( $1 \leq i, j \leq n$ ), which should be thought of as quantum analogues of the root vectors for  $(\mathfrak{gl}(n, \mathbb{C}), \mathfrak{h})$ . The  $L$ -operators are constructed as follows. Define elements  $E_{ij} \in U_q(\mathfrak{g})$  ( $1 \leq i \neq j \leq n$ ) inductively by the formulas

$$(4.3.22) \quad \begin{aligned} E_{i,i+1} &= X_i^+, \quad E_{ik} = E_{ij} E_{jk} - q E_{jk} E_{ij}, \\ E_{i+1,i} &= X_i^-, \quad E_{ki} = E_{kj} E_{ji} - q^{-1} E_{ji} E_{kj}, \end{aligned}$$

where  $1 \leq i < j < k \leq n$ . The  $L$ -operators  $L_{ij}^\pm$  are now defined by the formulas

$$(4.3.23) \quad \begin{aligned} L_{ii}^+ &= q^{\tilde{\varepsilon}_i}, \quad L_{ij}^+ = (q - q^{-1}) q^{\tilde{\varepsilon}_i} E_{ji}, \quad L_{ji}^+ = 0, \\ L_{ii}^- &= q^{-\tilde{\varepsilon}_i}, \quad L_{ji}^- = -(q - q^{-1}) E_{ij} q^{-\tilde{\varepsilon}_i}, \quad L_{ij}^- = 0 \end{aligned}$$

for  $1 \leq i < j \leq n$ . The  $L_{ij}^\pm$  generate  $U_q(\mathfrak{g})$  and satisfy commutation relations which can be expressed by means of the matrix  $R$  (4.3.2). Indeed, set  $L^\pm := \sum_{i,j} e_{ij} \otimes L_{ij}^\pm$ , where  $e_{ij} \in \text{End}(V)$  is regarded as the matrix units with respect to the fixed chosen basis  $\{v_i\}_{i=1}^n$  of  $V$ . Write  $R^+ := P R P$ , where  $P := \sum_{i,j} e_{ij} \otimes e_{ji}$  is the permutation operator. Then we have the commutation relations

$$R^+ L_1^\pm L_2^\pm = L_2^\pm L_1^\pm R^+, \quad R^+ L_1^+ L_2^- = L_2^- L_1^+ R^+.$$

Furthermore, the action of the comultiplication, counit and  $*$ -structure on the  $L$ -operators are given by

$$(4.3.24) \quad \Delta(L_{ij}^\pm) = \sum_k L_{ik}^\pm \otimes L_{kj}^\pm, \quad \varepsilon(L_{ij}^\pm) = \delta_{ij}, \quad (L_{ij}^\pm)^* = S(L_{ji}^\mp) \quad (1 \leq i, j \leq n).$$

The involution  $\tau = * \circ S: U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$  acts on the generators as

$$(4.3.25) \quad \tau(L_{ij}^\pm) = L_{ji}^\mp \quad (1 \leq i, j \leq n).$$

There is a natural Hopf  $*$ -algebra duality  $\langle \cdot, \cdot \rangle$  between  $U_q(\mathfrak{g})$  and  $A_q(U)$ . Writing  $T = \sum_{i,j} e_{ij} \otimes t_{ij}$ , the Hopf-algebra duality  $\langle \cdot, \cdot \rangle$  is determined by the formulas

$$(4.3.26) \quad \langle L_1^\pm, T_2 \rangle = R^\pm, \quad \langle L^\pm, \det_q \rangle = q^{\pm 1} \text{id}_V.$$

The relations satisfied by  $\langle \cdot, \cdot \rangle$  are explicitly given by

$$(4.3.27) \quad \begin{aligned} \langle XY, \phi \rangle &= \langle X \otimes Y, \Delta(\phi) \rangle, & \langle 1, \phi \rangle &= \varepsilon(\phi) \\ \langle X, \phi\psi \rangle &= \langle \Delta(X), \phi \otimes \psi \rangle, & \langle X, 1 \rangle &= \varepsilon(X), \\ \langle S(X), \phi \rangle &= \langle X, S(\phi) \rangle, \\ \langle X, \phi^* \rangle &= \overline{\langle \tau(X), \phi \rangle}, & \langle X^*, \phi \rangle &= \overline{\langle X, \tau(\phi) \rangle}, \end{aligned}$$

where  $X, Y \in U_q(\mathfrak{g})$ ,  $\phi, \psi \in A_q(U)$  and where  $\langle X \otimes Y, \phi \otimes \psi \rangle := \langle X, \phi \rangle \langle Y, \psi \rangle$ .

Write  $U_q(\mathfrak{h})$  for the subalgebra generated by the  $q^h$  ( $h \in P$ ). It is Laurent polynomial in the elements  $q^{\tilde{\varepsilon}^i}$  ( $1 \leq i \leq n$ ), and may be considered as the (quantized) algebra of functions on the Cartan subalgebra  $\mathfrak{h}$ . There is an induced Hopf  $*$ -algebra duality between  $U_q(\mathfrak{h})$  and  $A(T)$  such that

$$\langle q^h, z^\lambda \rangle := q^{\langle h, \lambda \rangle}, \quad z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n} \quad (h, \lambda \in P).$$

For a right  $A_q(U)$ -comodule  $(R, M)$ , the  $A_q(U)$ -coaction  $R$  can be differentiated using the Hopf-algebra pairing  $\langle \cdot, \cdot \rangle$ . This then yields a left  $U_q(\mathfrak{g})$ -module structure on  $M$  (cf. [96]). To be precise, the left  $U_q(\mathfrak{g})$ -action on  $M$  is defined by

$$(4.3.28) \quad X.m := \sum_{(m)} \langle X, m_{(2)} \rangle m_{(1)}, \quad (X \in U_q(\mathfrak{g}), m \in M),$$

where  $R(m) =: \sum_{(m)} m_{(1)} \otimes m_{(2)} \in M \otimes A_q(U)$  for  $m \in M$ . For example, by differentiating the vector corepresentation (4.3.10) we obtain the algebra homomorphism  $\rho_V : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$  which is uniquely determined by the formulas

$$(4.3.29) \quad R^\pm = \sum_{ij} e_{ij} \otimes \rho_V(L_{ij}^\pm), \quad (R^\pm)^{-1} = \sum_{ij} e_{ij} \otimes \rho_V(S(L_{ij}^\pm)),$$

where  $R^- := R^{-1}$ . By differentiation of right  $A_q(U)$ -coactions, a 1-1 correspondence is obtained between right  $A_q(U)$ -comodule structures on a finite dimensional vector space  $M$  and  $P$ -weighted left  $U_q(\mathfrak{g})$ -module structures on  $M$ . Recall that  $M$  is  $P$ -weighted if it is spanned by vectors that transform under  $U_q(\mathfrak{h})$  according to  $q^h \cdot v = q^{\langle h, \lambda \rangle} v$  ( $\lambda \in P$ ). A highest weight vector  $v$  of highest weight  $\lambda$  in a left  $U_q(\mathfrak{g})$ -module  $M$  is then characterized by the conditions  $L_{ij}^- \cdot v = 0$  ( $i > j$ ) (or, equivalently,  $X_i^+ v = 0$  for  $i \in [1, n-1]$ ) and  $q^h \cdot v = q^{\langle h, \lambda \rangle} v$ . There is a similar relationship between left  $A_q(U)$ -comodules and right  $U_q(\mathfrak{g})$ -modules. For a right  $U_q(\mathfrak{g})$ -module  $M$ , a weight vector  $0 \neq v \in M$  is a highest weight vector if  $v.L_{ij}^+ = 0$  ( $i < j$ ) (or, equivalently,  $v.X_i^- = 0$  for  $i \in [1, n-1]$ ).

**REMARK 4.3.2.** For a right  $A_q(U)$ -comodule  $(R, M)$  and for an element  $m \in M$ , we write  $m^\circ$  if we consider  $m$  as element in the left  $A_q(U)$ -comodule  $(R^\circ, M^\circ)$  (cf. Remark 4.3.1). Then, the differentiated right  $U_q(\mathfrak{g})$ -module structure on  $M^\circ$  is related to the differentiated left  $U_q(\mathfrak{g})$ -module structure on  $M$  by  $m^\circ.X = (X^*.m)^\circ$ , where  $m \in M$  and  $X \in U_q(\mathfrak{g})$ .

The coalgebra structure of  $A_q(U)$  induces in a natural way a  $A_q(U)$ -bicomodule structure on  $A_q(U)$ . By differentiating this  $A_q(U)$ -bicomodule structure,  $A_q(U)$  becomes a  $U_q(\mathfrak{g})$ -bimodule with  $U_q(\mathfrak{g})$ -symmetry. The action of the  $L$ -operators is then given by

$$(4.3.30) \quad L_1^\pm \cdot T_2 = T_2 R^\pm, \quad T_2 \cdot L_1^\pm = R^\pm T_2.$$

The irreducible decomposition of the  $U_q(\mathfrak{g})$ -bimodule  $A_q(U)$  is given by (4.3.9). This decomposition may also be characterized as the simultaneous eigenspace decomposition of  $A_q(U)$  under the action of the center  $\mathcal{Z} \subset U_q(\mathfrak{g})$ .

It can be shown that the pairing  $\langle \cdot, \cdot \rangle$  (4.3.27) is doubly non-degenerate. In particular,  $A_q(U)$  can be embedded as Hopf- $*$ -algebra into the Hopf- $*$ -algebra dual of  $U_q(\mathfrak{g})$ . The image under this embedding is the Hopf-subalgebra spanned by matrix elements of finite dimensional  $P$ -weighted  $U_q(\mathfrak{g})$ -modules (cf. Chapter 5).

#### 4.4. Spherical corepresentations

We call  $A_q(K) := A_q(U(n-l)) \otimes A_q(U(l))$  the quantized coordinate ring of the subgroup  $K = U(n-l) \times U(l)$  of  $U = U(n)$ . Corresponding to the embedding (4.2.3) there is an obvious surjective Hopf  $*$ -algebra morphism  $\pi_K : A_q(U(n)) \rightarrow A_q(K)$ . Write  $A_q(U/K)$  for the right  $A_q(K)$ -fixed elements in  $A_q(U)$ , i.e.

$$(4.4.1) \quad A_q(U/K) := \{\phi \in A_q(U) \mid (\text{id} \otimes \pi_K) \circ \Delta(\phi) = \phi \otimes 1\}.$$

Observe that  $A_q(U/K)$  is a left  $A_q(U)$ -comodule  $*$ -subalgebra of  $A_q(U)$ . The algebra  $A_q(U/K)$  can be interpreted as the quantized algebra of functions on the complex Grassmannian  $U/K$ .

For the study of  $A_q(U/K)$  it is important to obtain explicit information about  $A_q(K)$ -spherical corepresentations of  $A_q(U)$ , i.e. finite dimensional right  $A_q(U)$ -comodules with non-zero  $A_q(K)$ -fixed vectors. Recall that a vector  $v$  in a right  $A_q(U)$ -comodule  $M$  with comodule mapping  $R : M \rightarrow M \otimes A_q(U)$  is  $A_q(K)$ -fixed if

$$(4.4.2) \quad (\text{id} \otimes \pi_K) \circ R(v) = v \otimes 1.$$

One defines  $A_q(K)$ -fixed vectors in left  $A_q(U)$ -comodules in a similar way. This section is devoted to a proof of the following theorem on  $A_q(K)$ -spherical representations.

**THEOREM 4.4.1.** *Every finite dimensional irreducible corepresentation of  $A_q(U)$  has at most one  $A_q(K)$ -fixed vector (up to scalar multiples). The finite dimensional corepresentations with non-zero  $A_q(K)$ -fixed vectors are parametrized by the classical sublattice  $P_K^+$  of spherical dominant weights (cf. Section 4.2).*

**REMARK 4.4.2.** Let  $M$  be a finite dimensional right  $A_q(U)$ -comodule. It follows from Remark 4.3.1 that a vector  $v \in M$  is  $A_q(K)$ -fixed if and only if  $v \in M^\circ$  is  $A_q(K)$ -fixed. Hence, any statement about  $A_q(K)$ -fixed vectors in right  $A_q(U)$ -comodules immediately translates to a corresponding statement for left  $A_q(U)$ -comodules and vice-versa.

For the proof of Theorem 4.4.1 it suffices to show that the irreducible decomposition of  $V_R(\lambda)$  as a right  $A_q(K)$ -comodule is the same as the decomposition of the irreducible finite dimensional  $U(n)$ -highest weight representation of highest weight  $\lambda$ , decomposed as representation of the subgroup  $K$ . One way of establishing this result is by differentiating the coaction of  $A_q(U)$  on  $V_R(\lambda)$  using the doubly non degenerate Hopf-algebra pairing between  $A_q(U)$  and  $U_q(\mathfrak{g})$ . Then, the desired result follows from well-known results on the representation theory of quantized universal enveloping algebras. This approach is quite general, and will be treated in more detail in Chapter 6.

In this section another proof of Theorem 4.4.1 is given which does not use the quantized universal enveloping algebra. The strategy will be to relate the decomposition of the restriction to  $A_q(K)$  of the right  $A_q(U)$ -comodule  $V_R(\lambda)$  ( $\lambda \in P^+$ ) to characters on the maximal torus  $T$ . The following general result about corepresentation theory of semisimple coalgebras is needed (a coalgebra is said to be semisimple if every finite dimensional  $A$ -comodule is completely reducible).

**PROPOSITION 4.4.3.** *Let  $A$  and  $B$  be semisimple coalgebras. Then every finite dimensional  $A \otimes B$ -comodule is completely reducible. Denote  $\{V_\alpha \mid \alpha \in \hat{A}\}$  and  $\{V_\beta \mid \beta \in \hat{B}\}$  for a complete set of mutually inequivalent, irreducible, finite dimensional right  $A$  and  $B$ -comodules, respectively. Then*

$$(4.4.3) \quad \{V_\alpha \boxtimes V_\beta \mid \alpha \in \hat{A}, \beta \in \hat{B}\}$$

*is a complete set of mutually inequivalent, irreducible, finite dimensional right  $A \otimes B$ -comodules. Here  $V_\alpha \boxtimes V_\beta = V_\alpha \otimes V_\beta$  as a linear space and it has right comodule structure given by  $R_\alpha \boxtimes R_\beta := \sigma_{2,3} \circ (R_\alpha \otimes R_\beta)$ , where  $\sigma_{2,3}$  is the flip of the second and third tensor component and where  $R_\alpha$  and  $R_\beta$  are the right comodule mapping of  $V_\alpha$  and  $V_\beta$ , respectively.*

**PROOF.** It is known that  $V_\alpha \boxtimes V_\beta$  is an irreducible right  $A \otimes B$ -comodule for  $\alpha \in \hat{A}, \beta \in \hat{B}$  (cf. [67]). The right  $A$  and  $B$ -comodule structure on the first and second tensor component of  $V_\alpha \boxtimes V_\beta$  can be recovered by the coactions

$$(4.4.4) \quad R_1^\alpha := (\text{id}_{V_\alpha} \otimes \text{id}_{V_\beta} \otimes \text{id}_A \otimes \varepsilon_B) \circ (R_\alpha \boxtimes R_\beta),$$

respectively

$$(4.4.5) \quad R_2^\beta := (\text{id}_{V_\alpha} \otimes \text{id}_{V_\beta} \otimes \varepsilon_A \otimes \text{id}_B) \circ (R_\alpha \boxtimes R_\beta)$$

where  $\varepsilon_A$  and  $\varepsilon_B$  are the counits of  $A$  and  $B$ , respectively. So if  $V_\alpha \boxtimes V_\beta$  is isomorphic to  $V_{\alpha'} \boxtimes V_{\beta'}$  as  $A \otimes B$ -comodules, then they are isomorphic as  $A$  and as  $B$ -comodules (with coactions given by  $R_1^\alpha$  respectively  $R_2^\beta$ ), which can only happen when  $\alpha = \alpha'$  and  $\beta = \beta'$ . So (4.4.3) consists of mutually inequivalent, irreducible  $A \otimes B$ -comodules. It remains to prove that an arbitrary finite dimensional right  $A \otimes B$ -comodule  $(R, V)$  decomposes as a direct sum of irreducibles, each irreducible component being isomorphic to  $V_\alpha \boxtimes V_\beta$  for some  $\alpha \in \hat{A}, \beta \in \hat{B}$ .

Let  $R_A$  and  $R_B$  be the associated right  $A$  and  $B$ -comodule structures on  $V$  for the finite dimension right  $A \otimes B$ -comodule  $(R, V)$ :

$$R_A := (\text{id}_V \otimes \text{id}_A \otimes \varepsilon_B) \circ R, \quad R_B := (\text{id}_V \otimes \varepsilon_A \otimes \text{id}_B) \circ R.$$

Observe that

$$(4.4.6) \quad (R_A \otimes \text{id}_B) \circ R_B = R, \quad (R_B \otimes \text{id}_A) \circ R_A = (\text{id}_V \otimes \sigma_{A,B}) \circ R,$$

where  $\sigma_{A,B} : A \otimes B \rightarrow B \otimes A$  is the flip. When  $V$  is considered as corepresentation space of  $B$  via the coaction  $R_B$ , we write  $V^{(B)}$  for  $V$ . For  $\beta \in \hat{B}$ , let  $\text{Hom}_B(V_\beta, V^{(B)})$  be the linear space of intertwiners between the  $B$ -comodules  $V_\beta$  and  $V^{(B)}$ :

$$\text{Hom}_B(V_\beta, V^{(B)}) := \{\phi \in \text{Hom}_{\mathbb{C}}(V_\beta, V^{(B)}) \mid R_B \circ \phi = (\phi \otimes \text{id}_B) \circ R_\beta\}.$$

Then, due to Schur's lemma for corepresentations [67], we have a well defined  $\mathbb{C}$ -linear bijection

$$(4.4.7) \quad d := \bigoplus_{\beta \in \hat{B}} d_\beta : \bigoplus_{\beta \in \hat{B}} (\text{Hom}_B(V_\beta, V^{(B)}) \otimes V_\beta) \rightarrow V,$$

given component wise by  $d_\beta(\phi_\beta \otimes v) := \phi_\beta(v)$  ( $v \in V_\beta$ ,  $\phi_\beta \in \text{Hom}_B(V_\beta, V^{(B)})$ ).

We give now a right  $A$ -comodule structure  $R_\beta^{(A)}$  on  $\text{Hom}_B(V_\beta, V^{(B)})$  such that  $d_\beta$  intertwines of the corresponding right  $A \otimes B$ -comodule action on  $\text{Hom}_B(V_\beta, V^{(B)}) \boxtimes V_\beta$  and the right  $A \otimes B$ -comodule action  $R$  on  $V$ . Since  $A$  is semisimple, this will complete the proof of the proposition.

Let  $W(A) \subseteq A$  be the finite dimensional subspace of matrix elements of the right  $A$ -comodule action  $R_A$  on  $V$ , and let  $\{a_i\}_{i \in I}$  be a linear basis for  $W(A)$ . Fix an intertwiner  $\phi \in \text{Hom}_B(V_\beta, V^{(B)})$ , then for every  $v \in V_\beta$  there exist unique elements  $\phi_i(v) \in V$  such that

$$R_A(\phi(v)) = \sum_{i \in I} \phi_i(v) \otimes a_i.$$

The maps  $\phi_i : V_\beta \rightarrow V$  ( $i \in I$ ) are linear, and it is easily checked that

$$\phi_i \in \text{Hom}_B(V_\beta, V^{(B)}) \quad (\forall i \in I)$$

using (4.4.6) and the definition of the  $\phi_i$ . The map

$$R_\beta^{(A)} : \text{Hom}_B(V_\beta, V^{(B)}) \rightarrow \text{Hom}_B(V_\beta, V^{(B)}) \otimes A,$$

given by

$$R_\beta^{(A)}(\phi) := \sum_{i \in I} \phi_i \otimes a_i, \quad \phi \in \text{Hom}_B(V_\beta, V^{(B)}),$$

defines a right  $A$ -comodule structure on  $\text{Hom}_B(V_\beta, V^{(B)})$ . Using (4.4.6) it follows that  $d_\beta$  is an intertwiner for the  $A \otimes B$ -comodule actions  $\sigma_{2,3} \circ (R_\beta^{(A)} \otimes R_\beta)$  and  $R$ , which completes the proof of the proposition.  $\square$

For  $\lambda \in P^+$  with  $\lambda_n \geq 0$ , define the Schur polynomial  $s_\lambda(z) \in A(T)$  by

$$s_\lambda(z) := \Delta(z)^{-1} \sum_{w \in \mathfrak{S}_n} (-1)^{l(w)} z^{w(\lambda+\rho)},$$

with  $\Delta(z) := \prod_{i < j} (z_i - z_j)$  the Vandermonde determinant. For arbitrary  $\lambda \in P^+$  with  $\lambda_n \geq -m$  ( $m \in \mathbb{Z}$ ) define  $s_\lambda(z) := z^{-(m^n)} s_{\lambda+(m^n)}(z) \in A(T)$ . Then the  $s_\lambda$  ( $\lambda \in P^+$ ) are well-defined and form a basis of the subalgebra  $A(T)^{\mathfrak{S}_n}$  of symmetric Laurent polynomials. Recall that the character of a finite dimensional  $A_q(U)$ -comodule  $M$  is defined by  $\chi_M := \sum_i \pi_{ii} \in A_q(U)$ , where the  $\pi_{ij} \in A_q(U)$  are the matrix coefficients of  $M$  with respect to a basis of  $M$ . The character  $\chi_M$  is independent of the particular choice of basis for  $M$ . As shown in [96, (3.22)], the character  $\chi_\lambda \in A_q(U)$  of the irreducible comodule  $V_R(\lambda)$  satisfies

$$(4.4.8) \quad (\chi_\lambda)|_T(z) = s_\lambda(z) \quad (\lambda \in P^+),$$

as in the classical case ( $q = 1$ ).

**PROPOSITION 4.4.4.** *Let  $\lambda \in P^+$ . The restriction of the  $A_q(U)$ -comodule  $V_R(\lambda)$  to  $A_q(K)$  decomposes as*

$$(4.4.9) \quad V_R(\lambda) \simeq \bigoplus_{\mu, \nu} (V_R(\mu) \boxtimes V_R(\nu))^{\oplus c_{\mu, \nu}^\lambda},$$

the sum ranging over  $\mu \in P_{n-l}^+$ ,  $\nu \in P_l^+$ . Here the  $c_{\mu, \nu}^\lambda$  are the non-negative integers characterized by

$$(4.4.10) \quad s_\lambda(z_1, \dots, z_n) = \sum_{\mu, \nu} c_{\mu, \nu}^\lambda s_\mu(z_1, \dots, z_{n-l}) s_\nu(z_{n-l+1}, \dots, z_n),$$

the sum ranging over  $\mu \in P_{n-l}^+$ ,  $\nu \in P_l^+$ .

**PROOF.** There exists a decomposition (4.4.9) for certain uniquely determined non-negative integers  $c_{\mu, \nu}^\lambda$  by the previous proposition. It follows from (4.4.8) that the  $c_{\mu, \nu}^\lambda$  satisfy (4.4.10), since  $\chi_{M \boxtimes N} = \chi_M \otimes \chi_N \in A_q(K)$  for a finite dimensional right  $A_q(U(n-l))$ -comodule  $M$  and a finite dimensional right  $A_q(U(l))$ -comodule  $N$ .  $\square$

It follows from Proposition 4.4.4 that the abstract decomposition of an arbitrary finite dimensional right  $A_q(U)$ -comodule  $M$  into irreducible  $A_q(K)$ -comodules is the same as in the classical ( $q = 1$ ) case. Hence, Theorem 4.4.1 is now a consequence of Theorem 4.2.1.

**REMARK 4.4.5.** The proof of Theorem 4.4.1 can also be derived from Proposition 4.4.4 using the Littlewood-Richardson rule, which is a combinatorial rule for computing the coefficients  $c_{\mu, \nu}^\lambda$  in (4.4.10).

#### 4.5. A one parameter family of quantum Grassmannians

In this section a family of quantum Grassmannians is defined which depends on one real parameter  $-\infty < \sigma < \infty$  (cf. [92, Section 2]). The key ingredient in the definition will be the  $n \times n$  complex matrix  $J^\sigma$  defined by

$$(4.5.1) \quad J^\sigma := \sum_{1 \leq k \leq l} (1 - q^{2\sigma}) e_{kk} + \sum_{l < k < l'} e_{kk} - \sum_{1 \leq k \leq l} q^\sigma e_{kk'} - \sum_{1 \leq k \leq l} q^\sigma e_{k'l},$$

where  $k' := n - k + 1$  ( $1 \leq k \leq n$ ). Observe that  $\lim_{\sigma \rightarrow \infty} J^\sigma = J^\infty$ , where  $J^\infty$  is defined by

$$(4.5.2) \quad J^\infty := \sum_{k=1}^{n-l} e_{kk}$$

The subspace  $\mathfrak{k}^\sigma \subset U_q(\mathfrak{g})$  is by definition spanned by the coefficients of the matrix

$$(4.5.3) \quad L^+ J^\sigma - J^\sigma L^- \in \text{End}(V) \otimes U_q(\mathfrak{g}).$$

It follows from (4.3.24) that  $\mathfrak{k}^\sigma$  is a two-sided coideal in  $U_q(\mathfrak{g})$ , i.e.  $\Delta(\mathfrak{k}^\sigma) \subset U_q(\mathfrak{g}) \otimes \mathfrak{k}^\sigma + \mathfrak{k}^\sigma \otimes U_q(\mathfrak{g})$  and  $\varepsilon(\mathfrak{k}^\sigma) = 0$ . This is in fact true when  $J^\sigma$  is replaced by any  $n \times n$  matrix  $J$  in the definition of  $\mathfrak{k}^\sigma$ . Moreover, since  $J^\sigma$  is a symmetric matrix, it follows from (4.3.25) that  $\mathfrak{k}^\sigma$  is  $\tau$ -invariant.

Define the subalgebra  $A_q(\mathfrak{k}^\sigma \setminus U) \subset A_q(U)$  as the subspace of all left  $\mathfrak{k}^\sigma$ -invariant elements in  $A_q(U)$ , i.e. all  $a \in A_q(U)$  such that  $\mathfrak{k}^\sigma \cdot a = 0$ . As is well-known (cf. for instance [22, Proposition 1.9]), the fact that  $\mathfrak{k}^\sigma$  is a  $\tau$ -invariant two-sided coideal implies that  $A_q(\mathfrak{k}^\sigma \setminus U)$  is a  $*$ -subalgebra which is invariant under the right  $U_q(\mathfrak{g})$ -action on  $A_q(U)$  (or, equivalently, the left coaction of  $A_q(U)$  on itself). Important for the study of  $A_q(\mathfrak{k}^\sigma \setminus U)$  is the fact that  $X = J^\sigma$  is a solution of the reflection equation

$$(4.5.4) \quad R_{12} X_1 R_{12}^{-1} X_2 = X_2 R_{21}^{-1} X_1 R_{21},$$

where  $R_{12} := R$ ,  $R_{21} := PRP (= R^+)$ ,  $X_1 = X \otimes \text{id}_V$  and  $X_2 := \text{id}_V \otimes X$ . This fact can be verified by direct computations.

**REMARK 4.5.1.** The algebra  $A_q(\mathfrak{k}^\sigma \setminus U)$  can be considered as a quantized coordinate ring on the complex Grassmannian  $U(n)/(U(n-l) \times U(l))$  in the following way (see [92] for more details). The quantum function space on the  $n \times n$ -Hermitian matrices is defined as the algebra generated by  $X = (x_{ij})_{ij}$  with relations given by the reflection equation (4.5.4). It can be endowed with a  $*$ -structure and a left  $A_q(U)$ -coaction (the quantum analogue of the adjoint action). Since  $J^\sigma$  is a solution of (4.5.4) it gives rise to a ( $*$ -invariant) character of the quantum function algebra of Hermitian matrices. In other words,  $J^\sigma$  corresponds with a classical point in the quantum space of Hermitian matrices. Then  $A_q(\mathfrak{k}^\sigma \setminus U)$  may be considered as the quantized function algebra on the adjoint orbit of the classical point corresponding to  $J^\sigma$  (see [92, Proposition 2.4]). Since  $J^\sigma$  has two different eigenvalues 1 and  $-q^{2\sigma}$  with multiplicity  $n-l$  and  $l$  respectively, this quantum adjoint orbit is associated with the complex Grassmannian  $U(n)/(U(n-l) \times U(l))$ .



The quantized function algebra  $A_q(U/K)$  (4.4.1) can formally be interpreted as the algebra  $A_q(\mathfrak{k}^\sigma \setminus U)$  with  $\sigma \rightarrow \infty$ . To make this a little bit more explicit, we write

$$L^+ = \begin{pmatrix} {}^{11}L^+ & {}^{12}L^+ & {}^{13}L^+ \\ 0 & {}^{22}L^+ & {}^{23}L^+ \\ 0 & 0 & {}^{33}L^+ \end{pmatrix}, \quad L^- = \begin{pmatrix} {}^{11}L^- & 0 & 0 \\ {}^{21}L^- & {}^{22}L^- & 0 \\ {}^{31}L^- & {}^{32}L^- & {}^{33}L^- \end{pmatrix},$$

where  ${}^{11}L^+$  is an  $l \times l$  matrix,  ${}^{22}L^+$  an  $(n-2l) \times (n-2l)$  matrix etc. Let  $D$  be the  $l \times l$  matrix with 1's on the antidiagonal and 0's everywhere else. The coefficients of the matrix  $L^+ J^\sigma - J^\sigma L^-$  coincide with the coefficients of the following six matrices up to a sign:

$$(4.5.5) \quad \begin{aligned} (i) & \quad q^\sigma(D \cdot {}^{31}L^- - {}^{13}L^+ \cdot D) + (1 - q^{2\sigma})({}^{11}L^+ - {}^{11}L^-) \\ (ii) & \quad {}^{12}L^+ + q^\sigma D \cdot {}^{32}L^-, \\ (iii) & \quad {}^{23}L^+ \cdot q^\sigma D + {}^{21}L^-, \\ (iv) & \quad {}^{22}L^+ - {}^{22}L^-, \\ (v) & \quad {}^{11}L^+ \cdot q^\sigma D - q^\sigma D \cdot {}^{33}L^-, \\ (vi) & \quad {}^{33}L^+ \cdot q^\sigma D - q^\sigma D \cdot {}^{11}L^-. \end{aligned}$$

Obviously, the coefficients of the following matrix are also contained in  $\mathfrak{k}^\sigma$ :

$$(4.5.6) \quad (vii) \quad q^\sigma(D \cdot {}^{13}L^+ - {}^{31}L^- \cdot D) + (1 - q^{2\sigma})({}^{33}L^+ - {}^{33}L^-).$$

For later use, observe that the following elements of the ‘‘Cartan subalgebra’’  $U_q(\mathfrak{h})$  belong to  $\mathfrak{k}^\sigma$ :

$$(4.5.7) \quad L_{ii}^+ - L_{ii}^- \quad (l < i < l'), \quad L_{ii}^+ - L_{i'i'}^- \quad (1 \leq i \leq l), \quad L_{ii}^- - L_{i'i'}^+ \quad (1 \leq i \leq l).$$

It is clear from (4.5.5) and (4.5.6) that, in the limit  $\sigma \rightarrow \infty$ , the matrices in (i)–(vii) tend either to zero or to the following matrices

$$(4.5.8) \quad {}^{11}L^+ - {}^{11}L^-, \quad {}^{12}L^+, \quad {}^{21}L^-, \quad {}^{22}L^+ - {}^{22}L^-, \quad {}^{33}L^+ - {}^{33}L^-.$$

Again, the subspace  $\mathfrak{k}^\infty \subset U_q(\mathfrak{g})$  spanned by the coefficients of the matrices in (4.5.8) is a  $\tau$ -invariant two-sided coideal. Now, on the one hand,  $\mathfrak{k}^\infty$ -invariance in a left or right  $U_q(\mathfrak{g})$ -module  $M$  is obviously the same as invariance with respect to the Hopf  $*$ -subalgebra

$$U_q(\mathfrak{k}) := U_q(\mathfrak{gl}(n-l, \mathbb{C})) \otimes U_q(\mathfrak{gl}(l, \mathbb{C})) \hookrightarrow U_q(\mathfrak{gl}(n, \mathbb{C})),$$

where invariance of  $v \in M$  with respect to  $u \in U_q(\mathfrak{g})$  should be interpreted as  $u \cdot v = \varepsilon(u) \cdot v$  (if  $M$  is a left  $U_q(\mathfrak{g})$ -module). Using the Hopf-algebra duality between  $A_q(U)$  and  $U_q(\mathfrak{g})$  it can be easily shown that invariance of  $v \in M$  with respect to  $U_q(\mathfrak{k})$  is the same as invariance with respect to  $A_q(K)$  (cf. [22, Proposition 1.12]). It follows that  $A_q(K)$ -invariance is equivalent to  $\mathfrak{k}^\infty$ -invariance, hence  $A_q(\mathfrak{k}^\infty \setminus U) = A_q(U/K)$ .

REMARK 4.5.2. It should be observed that the matrix  $J^\infty$  also satisfies the reflection equation (4.5.4), but the subspace spanned by the coefficients of the matrix  $L^+ J^\infty - J^\infty L^-$  is strictly smaller than  $\mathfrak{k}^\infty$  and of little use for the purposes of this paper.

The following lemma is now a direct consequence of the arguments given above.

**LEMMA 4.5.3.** *Let  $M$  be a finite dimensional right  $A_q(U)$ -comodule with linear basis  $\{m_i\}_i$ . Consider  $M$  as left  $U_q(\mathfrak{g})$ -module using the differentiated action (4.3.28). Suppose that  $v_\sigma := \sum_i c_i(\sigma)m_i$  ( $c_i(\sigma) \in \mathbb{C}$ ) is a  $\mathfrak{k}^\sigma$ -fixed vector for all  $\sigma \in \mathbb{R}$  and that  $c_i := \lim_{\sigma \rightarrow \infty} c_i(\sigma)$  exists for all  $i$ . Then  $\sum_i c_i m_i$  is a  $A_q(K)$ -fixed vector in  $M$ .*

**REMARK 4.5.4.** In some suitable algebraic sense (cf. [11, Proposition 9.2.3]) the algebra  $U_q(\mathfrak{g})$  “tends” to  $U(\mathfrak{g})$  when  $q$  tends to 1. The corresponding limits of the  $L$ -operators are given by

$$L_{ij}^\pm / (q - q^{-1}) \rightarrow \pm e_{ji} \quad (i \leq j), \quad (q^{\varepsilon_i} - q^{-\varepsilon_i}) / (q - q^{-1}) \rightarrow e_{ii}$$

(cf. [91, (1.10), (1.11)]). Hence, by (4.5.8) respectively (4.5.5), the subspace  $\mathfrak{k}^\sigma \subset U_q(\mathfrak{g})$  ( $\sigma = \infty$  respectively  $\sigma = 0$ ) tends to the Lie subalgebra  $\mathfrak{k} = \mathfrak{gl}(n-l, \mathbb{C}) \oplus \mathfrak{gl}(l, \mathbb{C}) \subset \mathfrak{g}$  respectively  $\mathfrak{k}' \subset \mathfrak{g}$  (cf. Section 4.2) in the limit  $q \rightarrow 1$ .

Reflection equations play an important role in the quantization of symmetric spaces (cf. [91, Section 2], [94]). For the purposes of this chapter, the importance of this equation lies in the following fact. Recall that a vector  $w$  in a left  $U_q(\mathfrak{g})$ -module  $M$  is called  $\mathfrak{k}^\sigma$ -fixed if  $\mathfrak{k}^\sigma \cdot w = 0$  (a similar definition can be given for right  $U_q(\mathfrak{g})$ -modules).

**PROPOSITION 4.5.5.** ([95, Prop. 3.1], [92]) *Let  $J$  be any  $n \times n$  complex matrix. Write  $\mathfrak{k}^J \subset U_q(\mathfrak{g})$  for the two-sided coideal spanned by the coefficients of  $L^+ J - J L^-$ . The element*

$$w^J := \sum_{i,j} J_{ij} v_i \otimes v_j^* \in V \otimes V^*$$

*in the left  $U_q(\mathfrak{g})$ -module  $V \otimes V^*$  is a  $\mathfrak{k}^J$ -fixed vector if and only if  $J$  satisfies the reflection equation (4.5.4).*

**PROOF.** In the proof the same notational conventions as in [91, Proof of Proposition 2.3] will be used. Recall that the  $U_q(\mathfrak{g})$ -module structure on  $V^*$  corresponding to the dual  $A_q(U)$ -comodule  $V^*$  is given by

$$u \cdot v^*(v) := v^*(S(u) \cdot v) \quad (u \in U_q(\mathfrak{g}), v^* \in V^*, v \in V).$$

Set  $\mathbf{v} := (v_1, \dots, v_n)$ , then it follows from (4.3.29) that

$$(4.5.9) \quad L_1^\pm \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot R_{12}^\pm, \quad L_1^+ \cdot \mathbf{v}_2^* = \mathbf{v}_2^* \cdot (R_{21}^-)^{t_2}, \quad L_1^- \cdot \mathbf{v}_2^* = \mathbf{v}_2^* \cdot (R_{21}^+)^{t_2}.$$

Here  ${}^{t_2}$  denotes transposition with respect to the second tensor factor. An equation like  $L_1^+ \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot R_{12}^+$  should be interpreted as  $L_{ij}^+ \cdot v_k = \sum_{l=1}^n (R_{12}^+)_{jk}^{il} v_l$  for all  $1 \leq i, j, k \leq n$ , where  $R_{12}^+ = \sum_{i,j,k,l} (R_{12}^+)_{jl}^{ik} e_{ij} \otimes e_{kl}$ . Using the identities (4.5.9) one computes in shorthand notation,

$$\begin{aligned} L^+ J \cdot w^J &= L_1^+ \cdot (\mathbf{v}_2 J_2 \otimes (\mathbf{v}^*)_2^t) J_1 = (L_1^+ \cdot \mathbf{v}_2) J_2 \otimes L_1^+ \cdot (\mathbf{v}^*)_2^t J_1 \\ &= \mathbf{v}_2 R_{12}^+ J_2 R_{21}^- J_1 \otimes (\mathbf{v}^*)_2^t, \end{aligned}$$

since by (4.5.9) one has  $L_1^+ \cdot (\mathbf{v}^*)_2^t = R_{21}^- \cdot (\mathbf{v}^*)_2^t$ . On the other hand,

$$\begin{aligned} JL^- \cdot w^J &= J_1 L_1^- \cdot (\mathbf{v}_2 J_2 \otimes (\mathbf{v}^*)_2^t) = J_1 \mathbf{v}_2 R_{12}^- J_2 \otimes L_1^- \cdot (\mathbf{v}^*)_2^t \\ &= \mathbf{v}_2 J_1 R_{12}^- J_2 R_{21}^+ \otimes (\mathbf{v}^*)_2^t, \end{aligned}$$

since by (4.5.9) one has  $L_1^- \cdot (\mathbf{v}^*)_2^t = R_{21}^+ \cdot (\mathbf{v}^*)_2^t$ . It follows from the two preceding computations that  $w^J$  is  $\mathfrak{k}^J$ -fixed if and only if  $R_{12}^+ J_2 R_{21}^- J_1 = J_1 R_{12}^- J_2 R_{21}^+$ . Multiplying this last equation from the left and from the right by the permutation operator  $P$  gives (4.5.4), which proves the proposition.  $\square$

By Proposition 4.5.5 and the fact that the matrix  $J^\sigma$  satisfies the reflection equation (4.5.4), it follows that

$$(4.5.10) \quad w^\sigma := \sum_{ij} J_{ij}^\sigma v_i \otimes v_j^* \in V \otimes V^*$$

is a  $\mathfrak{k}^\sigma$ -fixed vector in the left  $U_q(\mathfrak{g})$ -module  $V \otimes V^*$ . Observe that  $\lim_{\sigma \rightarrow \infty} w^\sigma = w^\infty$ , with  $w^\infty$  the right  $A_q(U/K)$ -fixed vector defined in

$$(4.5.11) \quad w^\infty := \sum_{i,j} J_{ij}^\infty v_i \otimes v_j^* = \sum_{i=1}^{n-l} v_i \otimes v_i^*.$$

Since  $V \otimes V^* \simeq V(\varpi_1) \oplus V(0)$  as left  $U_q(\mathfrak{g})$ -modules (where  $V(0)$  is the trivial module) and since  $w^\sigma$  has a non-zero weight component of weight  $\varpi_1$ , it follows that  $V(\varpi_1)$  has a non-zero  $\mathfrak{k}^\sigma$ -fixed vector.

Next we construct a right  $\mathfrak{k}^\sigma$ -fixed vector in  $V^\circ \otimes (V^*)^\circ$ . Observe that a vector  $\tilde{w} = \sum_{i,j} \tilde{J}_{ij} v_i \otimes v_j^* \in V^\circ \otimes (V^*)^\circ$  for a real matrix  $\tilde{J} = \sum_{ij} \tilde{J}_{ij} e_{ij}$  is right  $\mathfrak{k}^\sigma$ -fixed if and only if  $\tilde{w}$  is left  $S(\mathfrak{k}^\sigma)$ -fixed as element in  $V \otimes V^*$  by the  $\tau$ -invariance of  $\mathfrak{k}^\sigma$  and by Remark 4.3.2. Reasoning as in the proof of Proposition 4.5.5, it follows that  $\tilde{w}$  is left  $S(\mathfrak{k}^\sigma)$ -fixed if  $\tilde{J}$  is a solution of the linear equation

$$(4.5.12) \quad J_1^\sigma (R_{21}^-)^{t_1} \tilde{J}_2 ((R_{21}^-)^{t_1})^{-1} = R^{t_1} \tilde{J}_2 (R^{t_1})^{-1} J_1^\sigma$$

where  $J^\sigma$  is given by (4.5.1). A solution  $\tilde{J} = \tilde{J}^\sigma$  of (4.5.12) is given by

$$(4.5.13) \quad \begin{aligned} \tilde{J}^\sigma := \sum_{1 \leq k \leq l} (1 - q^{2(n-2l)} q^{2\sigma}) e_{kk} + \sum_{l < k < l'} e_{kk} \\ - q^{\sigma-1} \sum_{1 \leq k \leq l} q^{2(k-l)} e_{kk'} - q^{\sigma-1} \sum_{1 \leq k \leq l} q^{2(k'-l)} e_{k'l}. \end{aligned}$$

We write  $\tilde{w}^\sigma = \sum_{ij} \tilde{J}_{ij}^\sigma v_i \otimes v_j^*$  for the corresponding right  $\mathfrak{k}^\sigma$ -fixed vector in  $V^\circ \otimes (V^*)^\circ$ . Similarly as was shown for left  $\mathfrak{k}^\sigma$ -fixed vectors it follows that  $V(\varpi_1)^\circ$  has a non-zero right  $\mathfrak{k}^\sigma$ -fixed vector. Observe that  $\lim_{\sigma \rightarrow \infty} \tilde{w}^\sigma = w^\infty$ , with  $w^\infty$  the  $A_q(K)$ -fixed vector given by (4.5.11).

Recall from the previous section that  $V(\lambda)$  has at most one  $\mathfrak{k}^\infty$ -fixed vector up to scalar multiples, and that  $V(\lambda)$  has non-zero  $\mathfrak{k}^\infty$ -fixed vectors if and only if  $\lambda \in P_K^+$

(cf. Theorem 4.4.1). We have the following analogous statement for  $\mathfrak{k}^\sigma$ -fixed vectors ( $-\infty < \sigma < \infty$ ).

**THEOREM 4.5.6.** ([92, Theorem 2.6]) *Let  $\lambda \in P^+$  and fix  $-\infty < \sigma < \infty$ . The irreducible left  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  with highest weight  $\lambda$  has at most one  $\mathfrak{k}^\sigma$ -fixed vector (up to scalar multiples). There exist non-zero  $\mathfrak{k}^\sigma$ -vectors in  $V(\lambda)$  if and only if  $\lambda \in P_K^+$ . The same statement holds for right  $\mathfrak{k}^\sigma$ -fixed vectors in  $V(\lambda)^\circ$ .*

In the remainder of this section a proof of Theorem 4.5.6 is given. Fix a parameter  $-\infty < \sigma < \infty$ . First of all, we have the following crucial lemma.

**LEMMA 4.5.7.** *Let  $\lambda \in P^+$  and fix  $-\infty < \sigma < \infty$ . Then any non-zero  $\mathfrak{k}^\sigma$ -fixed vector in the left  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  has a non-zero weight component of highest weight  $\lambda$ . The same statement holds for the right  $U_q(\mathfrak{g})$ -module  $V(\lambda)^\circ$ .*

The proof of the lemma follows by analyzing the particular form of the two-sided coideal  $\mathfrak{k}^\sigma$ . The details are omitted here, since the proof is analogous to the proof of [91, Lemma 3.2] and [24, Proposition 3.2].

Since the linear subspace of  $V(\lambda)$  (respectively  $V(\lambda)^\circ$ ) consisting of weight vectors of weight  $\lambda$  is one dimensional, it follows from Lemma 4.5.7 that every irreducible finite dimensional  $P$ -weighted  $U_q(\mathfrak{g})$ -module has at most one  $\mathfrak{k}^\sigma$ -fixed vector up to scalar multiples.

Set  $P_K = \bigoplus_{1 \leq r \leq l} \mathbb{Z} \varpi_r$ , where  $\varpi_r$  are the fundamental spherical weights (cf. Section 4.2). Observe that the assignment  $\lambda \mapsto \lambda^\natural$  as defined in Section 4.2 extends to an order-preserving bijection from  $P_K$  onto  $P_\Sigma$ . For  $\mu \in P_\Sigma$ , we write  $\lambda_\mu \in P_K$  for the inverse of  $\mu$  under the bijection  $\natural$ .

The following lemma is immediate from the fact that the Cartan type elements listed in (4.5.7) belong to  $\mathfrak{k}^\sigma$ .

**LEMMA 4.5.8.** *Let  $\lambda \in P_K^+$ ,  $-\infty < \sigma < \infty$  and assume that  $v \in V(\lambda)$  is a non-zero left  $\mathfrak{k}^\sigma$ -fixed vector. Let  $v = \sum_{\mu \leq \lambda} v_\mu$  be the decomposition of  $v$  in weight vectors, where  $v_\mu$  has weight  $\mu \in P$ . Then  $v_\mu = 0$  unless  $\mu \in P_K$ . The same statement is valid for the right  $U_q(\mathfrak{g})$ -module  $V(\lambda)^\circ$ .*

It follows from Lemma 4.5.7 and Lemma 4.5.8 that if  $V(\lambda)$  (respectively  $V(\lambda)^\circ$ ) has a non-zero  $\mathfrak{k}^\sigma$ -fixed vector, then  $\lambda \in P_K^+$ .

To finish the proof of Theorem 4.5.6 we have to show that all modules  $V(\lambda)$  and  $V(\lambda)^\circ$  ( $\lambda \in P_K^+$ ) have non-zero  $\mathfrak{k}^\sigma$ -fixed vectors. The existence of non-trivial  $\mathfrak{k}^\sigma$ -fixed vectors in  $V(\varpi_1)$  and in  $V(\varpi_1)^\circ$  is already proved. Explicit intertwining operators

$$\widehat{\Psi}_r: (V \otimes V^*)^{\otimes r} \rightarrow \Lambda_q^r(V) \otimes \Lambda_q^r(V^*), \quad (1 \leq r \leq l)$$

will be constructed to prove the existence of  $\mathfrak{k}^\sigma$ -fixed vectors in higher fundamental spherical representations. The proof of Theorem 4.5.6 is then completed by computing so-called principal term of  $\widehat{\Psi}_r((w^\sigma)^{\otimes r})$ , with  $w^\sigma \in V(\varpi_1)$  the  $\mathfrak{k}^\sigma$ -fixed vector given by (4.5.10).

Before giving the construction of  $\widehat{\Psi}_r$ , we first introduce the notion of principal term of a vector  $v \in \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  (cf. [95], [124]). For the present setting it is convenient to use a slightly modified definition of Noumi's and Sugitani's notion of principal term (cf. [95], [124]). The definition is based on certain specific properties of the comodule  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$ . The comodule  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  has a multiplicity free decomposition

$$(4.5.14) \quad \Lambda_q^r(V) \otimes \Lambda_q^r(V^*) \cong \bigoplus_{s=0}^r V(\varpi_s) \quad (1 \leq r \leq l)$$

as right  $A_q(U)$ -comodules, where  $\varpi_0 := 0 \in P_K^+$ . The decomposition (4.5.14) can be proved by computing the restriction of the character of the module  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  to the torus and using the classical Pieri formula for Schur functions [85, I, (5.17)] (cf. Proposition 4.4.4). Due to the multiplicity free decomposition (4.5.14), the module  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  is extremely useful for the study of  $\mathfrak{k}^\sigma$ -fixed vectors in  $V(\varpi_r)$ , as will be shown in the remainder of this chapter as well as in the next chapter. It follows from (4.5.14) that all the weights  $\mu \in P$  of the module  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  are  $\leq \varpi_r$  with respect to the dominance order. The vector  $v_{[1,r]} \otimes v_{[n-r+1,n]}^* \in \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  is the highest weight vector of the unique copy of  $V(\varpi_r)$  within  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$ . Suppose now that  $v = \sum_{\mu \leq \varpi_r} v_\mu$  is the weight space decomposition of a vector  $v \in \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$ , where  $v_\mu$  is the weight component of weight  $\mu \in P$ . Then the *principal term* of  $v$  is defined by

$$(4.5.15) \quad [v] := \sum_{\nu \in \mathcal{W}(1^r)} v_{\lambda_\nu}$$

(cf. [95], [124]), where  $\mathcal{W} = \mathcal{W}_l$  acts on  $(1^r) \in P_\Sigma^+ \subset P_\Sigma = \mathbb{Z}^l$  by permutations and inversions (cf. Section 4.2). Since  $\{\lambda_\nu \mid \nu \in \mathcal{W}(1^r)\}$  lies in the  $\mathfrak{S}_n$ -orbit of the highest weight  $\varpi_r \in P^+$  of  $V(\varpi_r)$ , it follows that the principal term of a vector  $v \in \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  lies in the unique copy of  $V(\varpi_r)$  within  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$ . If  $v$  is a non-zero  $\mathfrak{k}^\sigma$ -fixed vector in  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  and if  $v - [v]$  has a non-zero weight component of weight  $\nu$ , then, by Lemma 4.5.8,  $\nu \in P_K$  and  $\nu^\natural \in C(\varpi_r)$ , where

$$(4.5.16) \quad C(\mu) := \{\mu' \in P_\Sigma \mid w\mu' \prec \mu \ \forall w \in \mathcal{W}\} \quad (\mu \in P_\Sigma^+)$$

is the strict integral convex hull of  $\mathcal{W}\mu$ . In the next proposition the principal term of a  $\mathfrak{k}^\sigma$ -fixed vector in  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  (respectively in  $\Lambda_q^r(V)^\circ \otimes \Lambda_q^r(V^*)^\circ$ ) is compared with the elements  $u_r, \tilde{u}_r$  ( $1 \leq r \leq l$ ), which are defined by

$$(4.5.17) \quad u_r := \sum_{\substack{I \subset [1,l] \cup [l',n] \\ |I|=r, I \cap I' = \emptyset}} v_I \otimes v_{I'}^*, \quad \tilde{u}_r := \sum_{\substack{I \subset [1,l] \cup [l',n] \\ |I|=r, I \cap I' = \emptyset}} q^{(2\rho, \tilde{\varepsilon}_{I'})} v_I \otimes v_{I'}^*$$

where  $I' := \{i' \mid i \in I\}$ . The element  $u_r$  lies in the unique copy of  $V(\varpi_r)$  within  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$ , whereas  $\tilde{u}_r$  lies in the unique copy of  $V(\varpi_r)^\circ$  within  $\Lambda_q^r(V)^\circ \otimes \Lambda_q^r(V^*)^\circ$ . Observe that by the explicit form of the  $\mathfrak{k}^\sigma$ -fixed vectors  $w^\sigma$  respectively  $\tilde{w}^\sigma$ , we have

$$(4.5.18) \quad [w^\sigma] = -q^\sigma u_1, \quad [\tilde{w}^\sigma] = -q^{\sigma-1} q^{2(1-l)} \tilde{u}_1.$$

For the construction of the intertwiner  $\widehat{\Psi}_r$ , consider now the linear bijection  $\beta: V^* \otimes V \rightarrow V \otimes V^*$  determined by

$$(4.5.19) \quad \beta(v_i^* \otimes v_j) = q^{-\delta_{ij}} v_j \otimes v_i^* + (q^{-1} - q) \delta_{ij} \sum_{k < j} v_k \otimes v_k^*.$$

Write  $V_i := V, V_i^* := V^*$  ( $1 \leq i \leq r$ ). Define a linear bijection

$$\Psi_r: (V_1 \otimes V_1^*) \otimes \cdots \otimes (V_r \otimes V_r^*) \rightarrow (V_1 \otimes \cdots \otimes V_r) \otimes (V_1^* \otimes \cdots \otimes V_r^*)$$

by

$$(4.5.20) \quad \Psi_r := \beta_{1,r} \circ \beta_{2,r} \circ \cdots \circ \beta_{r-1,r} \circ \cdots \circ \beta_{13} \circ \beta_{23} \circ \beta_{12},$$

where  $\beta_{ij}$  acts by definition as the identity on all factors of the tensor product except for  $V_i^* \otimes V_j$ , on which it is equal to  $\beta$ . Write

$$\text{pr}_r: V^{\otimes r} \rightarrow \Lambda_q^r(V), \quad \text{pr}_r^*: (V^*)^{\otimes r} \rightarrow \Lambda_q^r(V^*)$$

for the canonical projections. We have now the following generalization of (4.5.18).

PROPOSITION 4.5.9. *Let  $1 \leq r \leq l$ . The operator*

$$\widehat{\Psi}_r: (V \otimes V^*)^{\otimes r} \rightarrow \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$$

defined by  $\widehat{\Psi}_r := (\text{pr}_r \otimes \text{pr}_r^*) \circ \Psi_r$  is a surjective intertwiner and

$$\begin{aligned} [\widehat{\Psi}_r((w^\sigma)^{\otimes r})] &= c_r(\sigma) u_r, & c_r(\sigma) &:= \left( \frac{q^\sigma}{q^2 - 1} \right)^r (q^2; q^2)_r, \\ [\widehat{\Psi}_r((\tilde{w}^\sigma)^{\otimes r})] &= \tilde{c}_r(\sigma) \tilde{u}_r, & \tilde{c}_r(\sigma) &:= \left( \frac{q^{\sigma-1} q^{2(1-l)}}{q^2 - 1} \right)^r (q^2; q^2)_r. \end{aligned}$$

Before giving a proof of Proposition 4.5.9, we first show how it implies Theorem 4.5.6. Since  $\mathfrak{k}^\sigma$  is a two-sided coideal and  $\Psi_r$  an intertwining operator, Proposition 4.5.9 shows that  $\Psi_r((w^\sigma)^{\otimes r})$  is a non-zero  $\mathfrak{k}^\sigma$ -fixed vector. Proposition 4.5.9 implies that the leading term of  $\Psi_r((w^\sigma)^{\otimes r})$  is non-zero, hence it follows that  $V(\varpi_r)$  ( $1 \leq r \leq l$ ) has a non-zero  $\mathfrak{k}^\sigma$ -fixed vector. Since any  $\lambda \in P_K^+$  can be written as a positive integral linear combination of the fundamental spherical weights  $\{\varpi_r\}_{1 \leq r \leq l}$ , it follows by an easy argument using tensor products and Lemma 4.5.7 that any  $\lambda \in P_K^+$  is actually spherical. For right  $\mathfrak{k}^\sigma$ -fixed vectors the same argument holds, since  $\widehat{\Psi}_r$  is also an intertwiner as map from the module  $(V^\circ \otimes (V^*)^\circ)^{\otimes r}$  to  $\Lambda_q^r(V)^\circ \otimes \Lambda_q^r(V^*)^\circ$  (cf. Remark 4.3.1).

So it remains to prove Proposition 4.5.9. The proof of this proposition proceeds by induction on  $r$ . The proof is broken up into a couple of lemmas.

LEMMA 4.5.10. *For  $2 \leq r \leq n + 1$  the linear mapping*

$$\widehat{\Phi}_r: \Lambda_q^{r-1}(V^*) \otimes V \rightarrow V \otimes \Lambda_q^{r-1}(V^*)$$

defined on the basis vectors  $v_I^* \otimes v_j$  ( $|I| = r - 1$ ,  $1 \leq j \leq n$ ) by

$$\widehat{\Phi}_r(v_I^* \otimes v_j) = \begin{cases} v_j \otimes v_I^* & \text{if } j \notin I, \\ q^{-1}v_j \otimes v_I^* - (q - q^{-1}) \sum_{m < j} \frac{s_q(I \setminus j; m)}{s_q(I \setminus j; j)} v_m \otimes v_{(I \setminus j) \cup m}^* & \text{if } j \in I \end{cases}$$

is an intertwining operator of right  $A_q(U)$ -comodules.

PROOF. Let  $P: V \otimes V \rightarrow V \otimes V$  denote the flip. Define a linear bijection  $\gamma: V \otimes V \rightarrow V \otimes V$  by  $\gamma := PR$ , with  $R$  as in (4.3.2). The action of  $\gamma$  on the basis vectors  $v_i \otimes v_j$  ( $1 \leq i, j \leq n$ ) is given by

$$(4.5.21) \quad \gamma(v_i \otimes v_j) = q^{\delta_{ij}} v_j \otimes v_i + (q - q^{-1}) \theta_{i,j} v_i \otimes v_j$$

with  $\theta_{i,j} := 1$  if  $i < j$  and  $\theta_{i,j} := 0$  otherwise. The fact that the commutation relations between the  $t_{ij} \in A_q(U)$  can be written as  $RT_1T_2 = T_2T_1R$  (cf. Section 4.3) implies that  $\gamma$  is an intertwining operator. Since  $R$  is a solution of the Quantum Yang-Baxter Equation,  $\gamma$  satisfies

$$(4.5.22) \quad \gamma_1 \circ \gamma_2 \circ \gamma_1 = \gamma_2 \circ \gamma_1 \circ \gamma_2,$$

with  $\gamma_i \in \text{End}(V^{\otimes 3})$  acting as  $\gamma$  on the  $i$ th and  $(i + 1)$ th tensor factors and as the identity on the remaining factor. Note furthermore that the exterior algebra  $\Lambda_q(V)$  is isomorphic as right  $A_q(U)$ -comodule algebra with  $T(V)/I$ , where  $T(V)$  is the tensor algebra of  $V$  and  $I \subset T(V)$  the two-sided ideal generated by  $\ker(\text{id} - q^{-1}\gamma) \subset V^{\otimes 2} \subset T(V)$ . Consider now the intertwiner  $\Gamma_k: V^{\otimes(k-1)} \otimes V \rightarrow V \otimes \Lambda_q^{k-1}(V)$  ( $2 \leq k \leq n + 1$ ) defined by

$$\Gamma_k = (\text{id} \otimes \text{pr}_{k-1}) \circ \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{k-1}.$$

Application of [35, Lemma 4.9 (1)] to the Yang-Baxter operator  $q^{-1}\gamma$  shows that there exists a unique bijective intertwiner

$$\widehat{\Gamma}_k: \Lambda_q^{k-1}(V) \otimes V \rightarrow V \otimes \Lambda_q^{k-1}(V)$$

such that  $\Gamma_k = \widehat{\Gamma}_k \circ (\text{pr}_{k-1} \otimes \text{id})$ . By a straightforward computation one verifies that

$$\begin{aligned} \widehat{\Gamma}_k(v_I \otimes v_j) &= q^{|I \cap j|} v_j \otimes v_I + \\ &\quad + (1 - q^2)(-q)^{-k+1} s_q(I; j) \sum_{\substack{i \in I \\ i < j}} s_q(i; I \setminus i) v_i \otimes v_{(I \setminus i) \cup j} \end{aligned}$$

for  $I \subset [1, n]$  with  $|I| = k - 1$  and  $1 \leq j \leq n$ .

Next, the linear mapping  $\delta_k: \Lambda_q^k(V^*) \rightarrow \Lambda_q^{n-k}(V) \otimes \mathbb{C} \det_q^{-1}$  ( $1 \leq k \leq n$ ) defined on the basis elements  $v_I^*$  ( $|I| = k$ ) by  $\delta_k(v_I^*) := s_q(I; I^c) v_{I^c} \otimes \det_q^{-1}$  is a bijective

intertwiner by (4.3.19). With the canonical identification  $V \otimes \mathbb{C} \det_q^{-1} \cong \mathbb{C} \det_q^{-1} \otimes V$  we have an intertwining operator  $\widehat{\Phi}_r: \Lambda_q^{r-1}(V^*) \otimes V \rightarrow V \otimes \Lambda_q^{r-1}(V^*)$  defined by

$$\widehat{\Phi}_r := q^{-1}(\text{id} \otimes \delta_{r-1}^{-1}) \circ (\widehat{\Gamma}_{n-r+2} \otimes \text{id}) \circ (\delta_{r-1} \otimes \text{id}).$$

Starting from the explicit expressions for  $\widehat{\Gamma}_{n-r+2}$  and  $\delta_{r-1}$ , a straightforward calculation shows that  $\widehat{\Phi}_r$  acts on the basis vectors  $v_I^* \otimes v_j$  as required.  $\square$

**COROLLARY 4.5.11.** *The linear mappings  $\beta$ ,  $\Psi_r$ , and  $\widehat{\Psi}_r$  are right  $A_q(U)$ -comodule homomorphisms.*

**PROOF.** The assertion follows from the previous lemma since  $\beta = \widehat{\Phi}_2$  and since the natural projections  $\text{pr}_r$  and  $\text{pr}_r^*$  intertwine the right  $A_q(U)$ -comodule actions.  $\square$

**LEMMA 4.5.12.** *Let  $1 \leq r \leq l$ . The bijective intertwining operator*

$$\Phi_r: (V_1^* \otimes \cdots \otimes V_{r-1}^*) \otimes V_1 \rightarrow V_1 \otimes (V_1^* \otimes \cdots \otimes V_{r-1}^*)$$

*defined by  $\Phi_r := \beta_{12} \circ \beta_{23} \circ \cdots \circ \beta_{r-2, r-1} \circ \beta_{r-1, r}$  satisfies*

$$(\text{id} \otimes \text{pr}_{r-1}^*) \circ \Phi_r = \widehat{\Phi}_r \circ (\text{pr}_{r-1}^* \otimes \text{id}).$$

**PROOF.** For  $I = \{i_1 < \cdots < i_r\} \subset [1, n]$ , set  $\tilde{v}_I^* := v_{i_r}^* \otimes \cdots \otimes v_{i_2}^* \otimes v_{i_1}^*$ . It is clear from the definitions that

$$(\text{id} \otimes \text{pr}_{r-1}^*) \circ \Phi_r(\tilde{v}_I^* \otimes v_j) = v_j \otimes v_I^* \quad \text{if } j \notin I.$$

If  $j \in I$ , then

$$(\text{id} \otimes \text{pr}_{r-1}^*) \circ \Phi_r(\tilde{v}_I^* \otimes v_j) = q^{-1} v_j \otimes v_I^* - (q - q^{-1}) \sum_{m < j} c(m, j) v_m \otimes v_{(I \setminus j) \cup m}^*,$$

where  $c(m, j) := (-q)^{|\{i \in I \mid m < i < j\}|}$  if  $m \notin I$ , and  $c(m, j) := 0$  otherwise. Using the definition of the  $q$ -signum  $s_q$ , it follows that  $c(m, j) = s_q(I \setminus j; m) s_q(I \setminus j; j)^{-1}$  if  $m < j$ , which concludes the proof of the lemma.  $\square$

Observe that the multiplication maps

$$\mu: \Lambda_q(V) \otimes \Lambda_q(V) \rightarrow \Lambda_q(V), \quad \mu^*: \Lambda_q(V^*) \otimes \Lambda_q(V^*) \rightarrow \Lambda_q(V^*)$$

are intertwiners of the  $A_q(U)$ -coactions, since  $\Lambda_q(V)$  and  $\Lambda_q(V^*)$  are  $A_q(U)$ -comodule algebras.

**LEMMA 4.5.13.** *The intertwining operator*

$$\widehat{\Theta}_r: \Lambda_q^{r-1}(V) \otimes \Lambda_q^{r-1}(V^*) \otimes V \otimes V^* \rightarrow \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$$



defined by  $\widehat{\Theta}_r := (\mu \otimes \mu^*) \circ (\text{id}_{\Lambda_q^{r-1}(V)} \otimes \widehat{\Phi}_r \otimes \text{id}_{V^*})$  satisfies

$$\begin{aligned} [\widehat{\Theta}_r(u_{r-1} \otimes w^\sigma)] &= -q^\sigma \frac{1 - q^{2r}}{1 - q^2} u_r \\ [\widehat{\Theta}_r(\tilde{u}_{r-1} \otimes \tilde{w}^\sigma)] &= -q^{\sigma-1} q^{2(1-l)} \frac{1 - q^{2r}}{1 - q^2} \tilde{u}_r \end{aligned}$$

for  $2 \leq r \leq l$ .

PROOF. If  $v$  is a vector of weight  $\mu$  in the domain of  $\widehat{\Theta}_r$ , then  $\widehat{\Theta}_r(v)$  is again a weight vector of weight  $\mu$  since  $\widehat{\Theta}_r$  intertwines the right  $A_q(U)$ -coaction. Hence, for a fixed  $I \subset [1, l] \cup [l', n]$  with  $I \cap I' = \emptyset$  and  $|I| = r - 1$ , we have that  $[\widehat{\Theta}_r(v_I \otimes v_{I'}^* \otimes v_s \otimes v_t^*)] = 0$  unless  $s, t \notin I \cup I'$  and  $s \neq t$ . By the explicit formulas for the action of  $\widehat{\Phi}_r$  (cf. Lemma 4.5.10), it follows that

$$[\widehat{\Theta}_r(u_{r-1} \otimes w^\sigma)] = -q^\sigma \sum_{I, k} v_I \wedge v_k \otimes v_{I'}^* \wedge v_{k'}^* = -q^\sigma \sum_J c_J v_J \otimes v_{J'}^*$$

where the first sum is taken over pairs  $(I, k)$  with  $I \subset [1, l] \cup [l', n]$ ,  $k \in [1, l] \cup [l', n]$ ,  $|I| = r - 1$ ,  $I \cap I' = \emptyset$  and  $k \notin I \cup I'$ , and the second sum is taken over subsets  $J \subset [1, l] \cup [l', n]$  with  $J \cap J' = \emptyset$  and  $|J| = r$ . The corresponding constant  $c_J$  is given by

$$c_J = \sum_{k \in J} s_q(J \setminus k; k) s_q(k'; J' \setminus k') = \sum_{k \in J} (s_q(J \setminus k; k))^2 = \sum_{s=0}^{r-1} q^{2s} = \frac{1 - q^{2r}}{1 - q^2}.$$

The proof for the leading term of  $\widehat{\Theta}_r(\tilde{u}_{r-1} \otimes \tilde{w}^\sigma)$  is similar.  $\square$

Proposition 4.5.9 can now be proved by induction to  $r$ , using the previous lemma for the induction step.

PROOF OF PROPOSITION 4.5.9. Define an intertwiner

$$\Theta_r : V^{\otimes(r-1)} \otimes (V^*)^{\otimes(r-1)} \otimes V \otimes V^* \rightarrow V^{\otimes r} \otimes (V^*)^{\otimes r}$$

by

$$\Theta_r := \text{id}_{V^{\otimes(r-1)}} \otimes \Phi_r \otimes \text{id}_{V^*}.$$

It follows from Lemma 4.5.12 that

$$(4.5.23) \quad (\text{pr}_r \otimes \text{pr}_r^*) \circ \Theta_r = \widehat{\Theta}_r \circ (\text{pr}_{r-1} \otimes \text{pr}_{r-1}^* \otimes \text{id}_V \otimes \text{id}_{V^*}).$$

From the definitions of  $\Psi_r$  and  $\Phi_r$  it follows that

$$\Psi_r = \Theta_r \circ (\Psi_{r-1} \otimes \text{id})$$

and hence by (4.5.23)

$$(4.5.24) \quad \widehat{\Psi}_r = \widehat{\Theta}_r \circ (\widehat{\Psi}_{r-1} \otimes \text{id}).$$

This allows us to prove the proposition by induction to  $r$ . The proposition is trivial for  $r = 1$ . Suppose that  $r \geq 2$ . By the induction hypotheses and Lemma 4.5.8 we have

$$\widehat{\Psi}_{r-1}((w^\sigma)^{\otimes r-1}) = c_{r-1}(\sigma)u_{r-1} + \sum_{\nu \in C((1^{r-1}))} v_{\lambda_\nu},$$

where  $v_{\lambda_\nu}$  is some weight vector of weight  $\lambda_\nu$  and  $C(\mu)$  is defined by (4.5.16). For  $\nu \in C((1^{r-1}))$  we have  $[\widehat{\Theta}_r(v_{\lambda_\nu} \otimes w^\sigma)] = 0$ , hence the induction step for the computation of  $[\widehat{\Psi}_r((w^\sigma)^{\otimes r})]$  follows by combining Lemma 4.5.13 with (4.5.24). The leading term  $[\widehat{\Psi}_r((\tilde{w}^\sigma)^{\otimes r})]$  can be computed in a similar way.  $\square$

**REMARK 4.5.14.** It should be observed that the proof of Theorem 4.5.6 differs in important details from the proof of Theorem 4.4.1. Observe for instance that Lemma 4.5.7 does not hold with  $\mathfrak{k}^\sigma$ -fixed replaced by  $\mathfrak{k}^\infty$ -fixed, since any  $\mathfrak{k}^\infty$ -fixed vector lies automatically in the zero weight space of the module.

#### 4.6. Zonal $(\sigma, \tau)$ -spherical functions

In this section the  $\mathfrak{k}^\tau$ -invariant  $(-\infty < \tau < \infty)$  functions are studied in the quantized coordinate ring  $A_q(\mathfrak{k}^\sigma \backslash U)$   $(-\infty < \sigma < \infty)$ . The results of this section were announced in [92, Section 3]. The rank 1 case of these results were earlier derived by Koornwinder [63] for  $n = 2$  and for arbitrary projective space by Noumi and Dijkhuizen [24].

Let  $-\infty < \sigma, \tau \leq \infty$  and denote  $\mathcal{H}^{\sigma, \tau}$  for the  $*$ -subalgebra of left  $\mathfrak{k}^\sigma$ -invariant and right  $\mathfrak{k}^\tau$ -invariant functions in  $A_q(U)$ . From Theorem 4.4.1, Theorem 4.5.6 and (4.3.9) we obtain the decomposition

$$(4.6.1) \quad \mathcal{H}^{\sigma, \tau} = \bigoplus_{\lambda \in P_K^+} \mathcal{H}^{\sigma, \tau}(\lambda), \quad \mathcal{H}^{\sigma, \tau}(\lambda) := W(\lambda) \cap \mathcal{H}^{\sigma, \tau},$$

the subspaces  $\mathcal{H}^{\sigma, \tau}(\lambda)$   $(\lambda \in P_K^+)$  being one dimensional. A non-zero element  $\varphi^{\sigma, \tau}(\lambda) \in \mathcal{H}^{\sigma, \tau}(\lambda)$  is called a zonal  $(\sigma, \tau)$ -spherical function. Since the decomposition (4.3.9) is orthogonal with respect to the inner product  $\langle \varphi, \psi \rangle = h(\psi^* \varphi)$ , the zonal spherical functions  $\varphi^{\sigma, \tau}(\lambda)$   $(\lambda \in P_K^+)$  are mutually orthogonal with respect to  $\langle \cdot, \cdot \rangle$ .

Let  $M$  denote a right  $A_q(U)$ -comodule with comodule mapping  $R_M$  and an invariant inner product  $\langle \cdot, \cdot \rangle$ . With any two elements  $v, w \in M$  we associate the matrix coefficient

$$(4.6.2) \quad \theta_M(v, w) := \sum_{(w)} \langle w_{(1)}, v \rangle w_{(2)} \in A_q(U), \quad R_M(w) := \sum_{(w)} w_{(1)} \otimes w_{(2)}.$$

The map  $\theta_M$  induces a linear map (denoted by the same symbol)

$$\theta_M : M^\circ \otimes M \rightarrow A_q(U),$$

which is surjective onto the subspace spanned by the matrix coefficients of  $M$ . If no confusion can arise we sometimes write  $\theta := \theta_M$ . The following lemma is a direct consequence of these definitions (cf. [91, Lemma 4.8]).

LEMMA 4.6.1. *Let  $M$  be a unitary right  $A_q(U)$ -comodule. The map  $\theta_M: M^\circ \otimes M \rightarrow A_q(U)$  satisfies the following properties:*

(i)  $\theta_M$  is a  $A_q(U)$ -bicomodule homomorphism, i.e.

$$(4.6.3) \quad \Delta \circ \theta_M = (\theta_M \otimes \text{id}) \circ (\text{id} \otimes R_M), \quad \Delta \circ \theta_M = (\text{id} \otimes \theta_M) \circ (R_M^\circ \otimes \text{id}),$$

where  $R_M^\circ$  is defined as in Remark 4.3.1.

(ii)  $\theta_M(v, w) = \tau(\theta_M(w, v))$  ( $v, w \in M$ ).

(iii) If  $M$  is irreducible of highest weight  $\lambda \in P^+$ , then  $\theta_M: M^\circ \otimes M \rightarrow W(\lambda)$  is an isomorphism of  $A_q(U)$ -bicomodules.

Lemma 4.6.1 can be used to construct zonal  $(\sigma, \tau)$ -spherical functions as follows. Let  $v_\sigma(\lambda) \in V(\lambda)$  respectively  $\tilde{v}_\tau(\lambda) \in V(\lambda)^\circ$  be a non-zero  $\mathfrak{k}^\sigma$ -fixed respectively  $\mathfrak{k}^\tau$ -fixed vector ( $\lambda \in P_K^+$ ). Let  $\langle \cdot, \cdot \rangle$  be a unitary inner product on  $V(\lambda)$ , and write  $\theta_\lambda$  for the map  $\theta$  in Lemma 4.6.1 with respect to the unitary comodule  $(V(\lambda), \langle \cdot, \cdot \rangle)$ . Then

$$(4.6.4) \quad \varphi^{\sigma, \tau}(\lambda) := \theta_\lambda(\tilde{v}_\tau(\lambda), v_\sigma(\lambda)) \in \mathcal{H}^{\sigma, \tau}(\lambda)$$

is a zonal  $(\sigma, \tau)$ -spherical function by Lemma 4.6.1. This leads to the following lemma.

LEMMA 4.6.2. *Let  $-\infty < \sigma, \tau < \infty$  and  $\lambda \in P_K^+$ . The image of  $\varphi^{\sigma, \tau}(\lambda)$  under the restriction map  $|_T: A_q(U) \rightarrow A(T)$  is of the form*

$$(4.6.5) \quad \varphi^{\sigma, \tau}(\lambda)|_T = c_{\lambda^\natural} m_{\lambda^\natural}(x) + \sum_{\nu \in C(\lambda^\natural)} c_\nu x^\nu, \quad c_\nu \in \mathbb{C},$$

with  $c_{\lambda^\natural} \neq 0$  and  $C(\nu)$  given by (4.5.16). Here the notation  $x^\nu := x_1^{\nu_1} x_2^{\nu_2} \dots x_l^{\nu_l}$  for  $\nu = (\nu_1, \dots, \nu_l) \in P_\Sigma$  is used, where the  $x_i$  ( $1 \leq i \leq l$ ) are defined by (4.2.10).

PROOF. Since any  $\lambda \in P_K^+$  can be written as a positive integral linear combination of the fundamental spherical weights  $\{\varpi_r\}_{1 \leq r \leq l}$ , it follows by an easy argument using tensor products, Lemma 4.5.7 and Lemma 4.5.8 that (4.6.5) for arbitrary  $\lambda \in P_K^+$  follows from (4.6.5) for the fundamental spherical weights  $\{\varpi_r\}_{r=1}^l$ .

So fix a fundamental weight  $\varpi_r$  ( $1 \leq r \leq l$ ). Consider the unitary inner product

$$(4.6.6) \quad \langle v_I \otimes v_J^*, v_K \otimes v_L^* \rangle = q^{-\langle 2\rho, \varepsilon_J \rangle} \delta_{I,K} \delta_{J,L}$$

on  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  (cf. Section 4.3) and write  $\theta$  for the map (4.6.2) associated with the module  $(\Lambda_q^r(V) \otimes \Lambda_q^r(V^*), \langle \cdot, \cdot \rangle)$ . By (4.5.14), the module  $V(\varpi_r)$  may be considered as irreducible component of  $\Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  with unitary structure given by the restriction of  $\langle \cdot, \cdot \rangle$  to  $V(\varpi_r)$ . Then, by Proposition 4.5.9 and the fact that  $u_r \in \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  (respectively  $\tilde{u}_r \in \Lambda_q^r(V)^\circ \otimes \Lambda_q^r(V^*)^\circ$ ) lies in the unique irreducible component  $V(\varpi_r)$  (respectively  $V(\varpi_r)^\circ$ ), the principal terms of the  $\mathfrak{k}^\sigma$ -fixed vector  $v_\sigma(\varpi_r)$  and the  $\mathfrak{k}^\tau$ -fixed vector  $\tilde{v}_\tau(\varpi_r)$  are given by

$$(4.6.7) \quad [v_\sigma(\varpi_r)] = c_r u_r, \quad [\tilde{v}_\tau(\varpi_r)] = \tilde{c}_r \tilde{u}_r$$

for non-zero constants  $c_r, \tilde{c}_r \in \mathbb{C}$ . For  $v_\mu \in \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  of weight  $\mu$  and  $\tilde{v}_\nu \in \Lambda_q^r(V)^\circ \otimes \Lambda_q^r(V^*)^\circ$  of weight  $\nu$  we have  $\theta(\tilde{v}_\nu, v_\mu)|_T = 0$  if  $\mu \neq \nu$ , and  $\theta(\tilde{v}_\nu, v_\mu)|_T$  is a

multiple of  $z^\mu$  if  $\mu = \nu$ . Using furthermore Lemma 4.5.8 and using the fact that  $\mathbb{C}[x^{\pm 1}]$  is the subalgebra of  $A(T)$  spanned by the monomials  $z^\mu (= x^{\mu^{\mathfrak{h}}})$  ( $\mu \in P_K$ ), we obtain from (4.6.7),

$$\begin{aligned} \varphi^{\sigma, \tau}(\lambda)|_T &= \theta(\tilde{v}_\tau(\varpi_r), v_\sigma(\varpi_r))|_T \\ &= \theta([\tilde{v}_\tau(\varpi_r)], [v_\sigma(\varpi_r)])|_T + \sum_{\nu \in C((1^r))} d_\nu x^\nu \\ &= d_{(1^r)} m_{(1^r)}(x) + \sum_{\nu \in C((1^r))} d_\nu x^\nu \end{aligned}$$

with  $d_{(1^r)} = c_r \tilde{c}_r \neq 0$ , since  $\theta(\tilde{u}_r, u_r)|_T = m_{(1^r)}(x)$ . This completes the proof of (4.6.5) for the fundamental spherical weights.  $\square$

Lemma 4.6.2 has the following important consequence.

**COROLLARY 4.6.3.** *The restriction of the map  $|_T: A_q(U) \rightarrow A(T)$  to  $\mathcal{H}^{\sigma, \tau}$  defines an injection from  $\mathcal{H}^{\sigma, \tau}$  into  $\mathbb{C}[x^{\pm 1}]$  for  $-\infty < \sigma, \tau < \infty$ . In particular,  $\mathcal{H}^{\sigma, \tau}$  is a commutative algebra for  $-\infty < \sigma, \tau < \infty$ .*

Recall from [104] and [92, Section 3] the Casimir operator

$$C := \sum_{ij} q^{2(n-i)} L_{ij}^+ S(L_{ji}^-) \in U_q(\mathfrak{g}).$$

Since  $C$  is central, it acts on  $W(\lambda)$  ( $\lambda \in P^+$ ) as a scalar  $\chi_\lambda(C)$ , which is given by

$$\chi_\lambda = \sum_{k=1}^n q^{2(\lambda_k + n - k)}.$$

Also,  $C$  maps  $\mathcal{H}^{\sigma, \tau}$  into itself. Therefore, if  $-\infty < \sigma, \tau < \infty$ , the restricted Casimir operator  $C: \mathcal{H}^{\sigma, \tau} \rightarrow \mathcal{H}^{\sigma, \tau}$  induces an operator

$$L: \mathcal{H}^{\sigma, \tau}|_T \rightarrow \mathcal{H}^{\sigma, \tau}|_T \subset \mathbb{C}[x^{\pm 1}],$$

which is called the radial part of  $C$ . Explicitly,  $L$  is the map which satisfies

$$L(\phi|_T) = (C\phi)|_T, \quad \forall \phi \in \mathcal{H}^{\sigma, \tau}.$$

Crucial for the identification of the zonal  $(\sigma, \tau)$ -spherical functions is the realization of the radial part  $L$  of the Casimir element  $C$  as the restriction to  $\mathcal{H}^{\sigma, \tau}|_T$  of an explicit second order  $q^2$ -difference operator on  $\mathbb{C}[x^{\pm 1}]$ . Without proof we will state here the result (see [92, Section 3]).

**THEOREM 4.6.4.** ([92]) *Let  $-\infty < \sigma, \tau < \infty$  and  $\lambda \in P_K^+$ . The operator  $L - \chi_\lambda(C)$  id coincides on  $\mathcal{H}^{\sigma, \tau}|_T \subset \mathbb{C}[x^{\pm 1}]$  with a constant multiple of Koornwinder's second order  $q^2$ -difference operator  $D - E_{\lambda^{\mathfrak{h}}}$  id in the variables  $x = (x_1, \dots, x_l)$  with base*

$q^2$  and parameters  $(\underline{t}, t) = (\underline{t}^{\sigma, \tau}, q^2)$ , given by

$$(4.6.8) \quad \begin{aligned} t_0^{\sigma, \tau} &= -q^{\sigma+\tau+1}, & t_1^{\sigma, \tau} &= -q^{-\sigma-\tau+1}, \\ t_2^{\sigma, \tau} &= q^{\sigma-\tau+1}, & t_3^{\sigma, \tau} &= q^{-\sigma+\tau+2(n-2l)+1}. \end{aligned}$$

For a proof of the theorem for rank 1, see [24]. In [94] a proof can be found for the special case  $n = 2l$  and  $\sigma = \tau = 0$ .

Note that  $\underline{t}^{\sigma, \tau} \in V_K$  for  $-\infty < \sigma, \tau < \infty$  (cf. Section 2.6) and that  $t_0^{\sigma, \tau}, t_1^{\sigma, \tau}, t_2^{\sigma, \tau}, t_3^{\sigma, \tau} \in (0, 1)$ . In particular, the eigenvalues  $E_{\lambda_{\mathfrak{h}}}$  are mutually different for compatible weights when  $-\infty < \sigma, \tau < \infty$  (see Proposition 3.6.5).

We write  $D_{\sigma, \tau}$  for Koornwinder's second order  $q^2$ -difference operator in base  $q^2$  with parameters  $(\underline{t}^{\sigma, \tau}, q^2)$ , and  $E_{\mu}^{\sigma, \tau}$  ( $\mu \in P_{\Sigma}^+$ ) for the corresponding eigenvalues. We furthermore write  $P_{\mu}^{\sigma, \tau}(x) := P_{\mu}(x; \underline{t}^{\sigma, \tau}; q^2)$  ( $\mu \in P_{\Sigma}^+$ ) for the corresponding monic Koornwinder polynomials in base  $q^2$ . By Theorem 4.6.4,  $\varphi^{\sigma, \tau}(\lambda)|_T \in \mathbb{C}[x^{\pm 1}]$  is an eigenfunction of  $D_{\sigma, \tau}$  with eigenvalue  $E_{\lambda_{\mathfrak{h}}}^{\sigma, \tau}$  for  $\lambda \in P_K^+$ . By [94, Lemma 6.2], any eigenfunction  $\phi(x) \in \mathbb{C}[x^{\pm 1}]$  of  $D_{\sigma, \tau}$  with eigenvalue  $E_{\mu}^{\sigma, \tau}$  ( $\mu \in P_{\Sigma}^+$ ) which is of the particular form

$$\phi(x) = c_{\mu} m_{\mu}(x) + \sum_{\nu \in C(\mu)} c_{\nu} x^{\nu}$$

is a constant multiple of the Koornwinder polynomial  $P_{\mu}^{\sigma, \tau}(x)$ . Combined with Lemma 4.6.2, the following main result of the paper [92] is obtained.

**THEOREM 4.6.5. ([92])** *Let  $-\infty < \sigma, \tau < \infty$ . The restriction  $\varphi^{\sigma, \tau}(\lambda)|_T$  of the zonal spherical function  $\varphi^{\sigma, \tau}(\lambda) \in \mathcal{H}^{\sigma, \tau}(\lambda)$  ( $\lambda \in P_K^+$ ) is equal to the Koornwinder polynomial  $P_{\lambda_{\mathfrak{h}}}^{\sigma, \tau}(x)$ , up to a non-zero scalar multiple. In particular,  $|_T$  defines an algebra isomorphism from  $\mathcal{H}^{\sigma, \tau}$  onto  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$ .*

**REMARK 4.6.6.** It should be observed here that the assumption  $-\infty < \sigma, \tau < \infty$  in the preceding arguments is absolutely essential. In fact, the map  $|_T: A_q(U) \rightarrow A(T)$  factors through the projection  $\pi_K: A_q(U) \rightarrow A_q(K)$ . This implies that the image of  $\mathcal{H}^{\sigma, \tau}$  under  $|_T$  is one dimensional as soon as either  $\sigma$  or  $\tau$  is infinite.

#### 4.7. Limit transitions on quantum Grassmannians

In this section the right  $\mathfrak{k}^{\tau}$ -invariant ( $-\infty < \tau \leq \infty$ ) functions in the quantized coordinate ring  $A_q(U/K) = A_q(\mathfrak{k}^{\infty} \backslash U)$  are studied. The harmonic analysis for  $\sigma = \infty$  and/or  $\tau = \infty$  will be derived as limit case of the harmonic analysis for  $-\infty < \sigma, \tau < \infty$ , using explicit knowledge of the limit transitions from Koornwinder polynomials to multivariable big and little  $q$ -Jacobi polynomials. The rank 1 case of these results were earlier derived by Koornwinder [63] for 2-spheres and for arbitrary projective space by Noumi and Dijkhuizen [24].

For the proper interpretation of the limit transitions of the zonal spherical functions, a careful study is needed of the pre-images of the  $\mathcal{W}$ -invariant functions  $e_s(x) := m_{(1^s)}(x)$

( $1 \leq s \leq l$ ) under the isomorphism  $|_T : \mathcal{H}^{\sigma, \tau} \rightarrow \mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$ . For  $-\infty < \sigma, \tau < \infty$ , write  $e_r^{\sigma, \tau}$  for the unique element in  $\mathcal{H}^{\sigma, \tau}$  such that its restriction to the torus is equal to  $e_r(x)$  ( $1 \leq r \leq l$ ). It is convenient to put  $e_0(x) := 1$  and  $e_0^{\sigma, \tau} := 1$ . The  $\mathcal{W}$ -invariant functions  $\{e_r\}_{r=1}^l$  are algebraically independent generators of  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$ . In other words, the assignment

$$(4.7.1) \quad \hat{P}(e_1(x), \dots, e_l(x)) := P(x), \quad P \in \mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$$

defines an algebra isomorphism  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}} \rightarrow \mathbb{C}[y]$ , where  $y = (y_1, \dots, y_l)$  are  $l$  independent variables. It follows now from Theorem 4.6.5 that the elements  $\{e_r^{\sigma, \tau}\}_{r=1}^l$  are algebraically independent generators of the algebra  $\mathcal{H}^{\sigma, \tau}$ .

Using Theorem 4.6.5 it is now easy to derive an explicit form of the restriction of the normalized Haar functional  $h$  to  $\mathcal{H}^{\sigma, \tau}$ . Recall that the parameters  $\underline{t}^{\sigma, \tau}$  lies in the parameter domain  $V_K$  for  $-\infty < \sigma, \tau < \infty$  (see Definition 2.6.1 for the definition of  $V_K$ ). Let  $\langle \cdot, \cdot \rangle_{\underline{t}^{\sigma, \tau}, q^2}$  be the non degenerate bilinear form for which the Koornwinder polynomials  $P_{\mu}^{\sigma, \tau}(x)$  ( $\mu \in P_{\Sigma}^+$ ) are mutually orthogonal (see Theorem 2.6.6). Write  $\langle \phi \rangle_{\sigma, \tau} := \langle \phi, 1 \rangle_{\underline{t}^{\sigma, \tau}, q^2}$  for the constant term of  $\phi \in \mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$ . Observe that  $\langle 1 \rangle_{\sigma, \tau} = \mathcal{N}(0; \underline{t}^{\sigma, \tau}; q^2)$  is known explicitly by Gustafson's evaluation of the multidimensional Askey-Wilson integral (cf. (2.1.1)). In particular, the constant term  $\langle 1 \rangle_{\sigma, \tau}$  is non-zero for all  $-\infty < \sigma, \tau < \infty$ .

**COROLLARY 4.7.1.** *Let  $-\infty < \sigma, \tau < \infty$ . The restricted Haar functional  $h : \mathcal{H}^{\sigma, \tau} \rightarrow \mathbb{C}$  is explicitly given by*

$$h(\hat{P}(e_1^{\sigma, \tau}, e_2^{\sigma, \tau}, \dots, e_l^{\sigma, \tau})) = \frac{\langle P \rangle_{\sigma, \tau}}{\langle 1 \rangle_{\sigma, \tau}} \quad (P \in \mathbb{C}[x^{\pm 1}]^{\mathcal{W}}).$$

**PROOF.** The left and the right hand side are equal to zero for  $P = P_{\mu}^{\sigma, \tau}$  with  $0 \neq \mu \in P_{\Sigma}^+$ , and equal to 1 for  $P = 1$ . The corollary follows now by linearity, since the Koornwinder polynomials  $P_{\mu}^{\sigma, \tau}(x)$  ( $\mu \in P_{\Sigma}^+$ ) form a linear basis of  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$ .  $\square$

Recall the intertwiner

$$\hat{\Psi}_r : (V \otimes V^*)^{\otimes r} \rightarrow \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$$

defined in Proposition 4.5.9. Introduce left  $\mathfrak{k}^{\sigma}$ -fixed vectors  $w_r^{\sigma} \in \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  and right  $\mathfrak{k}^{\tau}$ -fixed vectors  $\tilde{w}_r^{\tau} \in \Lambda_q^r(V)^{\circ} \otimes \Lambda_q^r(V^*)^{\circ}$  by

$$w_r^{\sigma} := \hat{\Psi}_r((w^{\sigma})^{\otimes r}), \quad \tilde{w}_r^{\tau} := \hat{\Psi}_r((\tilde{w}^{\tau})^{\otimes r}) \quad (1 \leq r \leq l).$$

Here we have used the notation  $\tilde{w}^{\infty} := w^{\infty}$  (4.5.11) in case of  $\tau = \infty$ , which is compatible with the definition of  $\tilde{w}^{\tau}$  for  $-\infty < \tau < \infty$  since  $\lim_{\tau \rightarrow \infty} \tilde{w}^{\tau} = w^{\infty}$ . Consider now the  $(\sigma, \tau)$ -spherical elements

$$(4.7.2) \quad \varphi_r^{\sigma, \tau} := \theta(\tilde{w}_r^{\tau}, w_r^{\sigma}) \in \bigoplus_{s=0}^r \mathcal{H}^{\sigma, \tau}(\varpi_s) \quad (1 \leq r \leq l)$$

(cf. (4.5.14)), where  $\theta$  is the map (4.6.2) associated with the unitary module  $(\Lambda_q^r(V) \otimes \Lambda_q^r(V^*), \langle \cdot, \cdot \rangle)$  (see (4.6.6) for the definition of  $\langle \cdot, \cdot \rangle$ ). It is convenient to put  $\varphi_0^{\sigma, \tau} := 1$ . By

Theorem 4.6.5 and (4.7.2),  $\varphi_r^{\sigma, \tau}|_T$  is a linear combination of the  $\mathcal{W}$ -invariant functions  $e_s(x) \in \mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$  ( $0 \leq s \leq r$ ).

LEMMA 4.7.2. *Let  $-\infty < \sigma, \tau < \infty$ ,  $1 \leq r \leq l$ . In the expansion*

$$\varphi_r^{\sigma, \tau}|_T = a_r^r(q^\sigma, q^\tau)e_r + \cdots + a_0^r(q^\sigma, q^\tau)e_0,$$

*each coefficient  $a_i^r$  is a polynomial in  $q^\sigma$  and  $q^\tau$  which is the sum of monomials of partial degree  $\geq i$  in each of the variables. Moreover,  $a_r^r(q^\sigma, q^\tau) = c_r(\sigma)\tilde{c}_r(\tau) = cq^{r\sigma+r\tau}$  with  $c \neq 0$  independent of  $q^\sigma$  and  $q^\tau$ , where  $c_r(\sigma)$  and  $\tilde{c}_r(\tau)$  are defined in Proposition 4.5.9.*

PROOF. It is obvious from the definitions that the coefficients are polynomial in  $q^\sigma$  and  $q^\tau$ . To prove the estimates on the partial degrees, we study the action of the intertwiner  $\widehat{\Psi}_r$  on the vectors  $(w^\sigma)^{\otimes r}$  and  $(\tilde{w}^\tau)^{\otimes r}$  in detail. We proceed in a number of steps.

1) Let  $1 \leq i_1 \leq \cdots \leq i_r \leq n$  and  $1 \leq j_1 \leq \cdots \leq j_r \leq n$  be integers. We use the shorthand notation  $\underline{i} := (i_1, \dots, i_r)$ ,  $\underline{j} := (j_1, \dots, j_r)$ . Call a tensor  $t$  in some tensor product space made up of factors  $V$  or  $V^*$  (the total number of factors  $V$  being equal to the total number of factors  $V^*$ ) a basic tensor of type  $(\underline{i}, \underline{j})$  if  $t$  is the tensor product in any given order of the vectors  $v_{i_1}, \dots, v_{i_r}$  and  $v_{j_1}^*, \dots, v_{j_r}^*$ . Let  $n_k(\underline{i})$  denote the cardinality of the set  $\{p \in [1, r] \mid i_p = k\}$ . For a basic tensor  $t$  of type  $(\underline{i}, \underline{j})$  define

$$n(t) := \sum_{k=1}^n \min(n_k(\underline{i}), n_k(\underline{j})).$$

From an informal point of view,  $n(t)$  is the number of factors  $v_i$  in  $t$  that “cancel” against a factor  $v_i^*$ . Recall the intertwiner  $\Psi_r: (V \otimes V^*)^{\otimes r} \rightarrow V^{\otimes r} \otimes (V^*)^{\otimes r}$  defined in (4.5.20). Let  $t$  be a basic tensor in  $(V \otimes V^*)^{\otimes r}$ . Since  $\Psi_r$  is the composition of the intertwiner  $\beta$  (see (4.5.20)) it follows by inspection of (4.5.19) that  $\Psi_r(t)$  is a linear combination of basic tensors  $t'$  in  $V^{\otimes r} \otimes (V^*)^{\otimes r}$  with  $n(t') = n(t)$ .

(2) A basic tensor  $t \in (V \otimes V^*)^{\otimes r}$  is called typical if it is a product of tensors in  $V \otimes V^*$  of type  $v_i \otimes v_i^*$  ( $1 \leq i \leq n-l$ ),  $v_i \otimes v_i^*$ ,  $v_i \otimes v_i^*$  ( $1 \leq i \leq l$ ). We call a typical tensor  $t \in (V \otimes V^*)^{\otimes r}$   $k$ -typical if the number of factors of type  $v_i \otimes v_i^*$  ( $i \in [1, l] \cup [l', n]$ ) is equal to  $k$ . If  $t$  is a  $k$ -typical tensor then  $\widehat{\Psi}_r(t)$  is a linear combination of elements  $v_I \otimes v_J^*$  where  $I, J \subset [1, n]$  are such that  $|I| = |J| = r$  and  $|I \cap J| \geq r - k$ . In fact, this follows from (1) and the definition of  $\widehat{\Psi}_r$ , since  $n(t) \geq r - k$ .

(3) It is an immediate consequence of the definition of the coactions on  $\Lambda_q^r(V)$  and  $\Lambda_q^r(V^*)$  and of (4.3.19) that

$$\theta(v_I \otimes v_J^*, v_K \otimes v_L^*)|_T = q^{-(2\rho, \tilde{\varepsilon}_J)} \delta_{I,K} \delta_{J,L} z^{\tilde{\varepsilon}_I - \tilde{\varepsilon}_J},$$

for  $I, J \subset [1, n]$  with  $|I| = |J| = r$ .

(4) Let  $t$  be a  $k$ -typical tensor and  $t'$  a  $m$ -typical tensor. Fix a weight  $\mu \in \mathcal{W}(1^i) \subset P_\Sigma$  ( $i \in [1, r]$ ) and suppose that the coefficient of  $z^{\lambda\mu}$  is non-zero in the expansion of  $\theta(\widehat{\Psi}_r(t), \widehat{\Psi}_r(t'))|_T$  with respect to the basis  $\{z^\lambda\}_{\lambda \in P}$  of  $A(T)$ . Then,  $k \geq i$  and  $m \geq i$ . This is a straightforward consequence of (2) and (3).

(5) There is a unique expansion  $(w^\sigma)^{\otimes r} = \sum_{k=0}^r \sum_{t_k} c_{t_k} t_k$  where  $t_k$  runs over all  $k$ -typical tensors in  $(V \otimes V^*)^{\otimes r}$  and where the  $c_{t_k} \neq 0$  are linear combinations of monomials  $(q^\sigma)^i$  with  $i \geq k$ . Similarly, there is a unique expansion  $(\tilde{w}^\tau)^{\otimes r} = \sum_{k=0}^r \sum_{t'_k} d_{t'_k} t'_k$  where  $t'_k$  runs over all  $k$ -typical tensors in  $(V \otimes V^*)^{\otimes r}$  and where the  $d_{t'_k} \neq 0$  are linear combinations of monomials  $(q^\tau)^i$  with  $i \geq k$ . Hence,

$$\varphi_r^{\sigma, \tau} |T = \sum_{k,m=0}^r \sum_{t_k, t'_m} c_{t_k t'_m} \theta(\widehat{\Psi}_r(t'_m), \widehat{\Psi}_r(t_k)) |T$$

with  $c_{t_k t'_m} = \overline{c_{t_k} d_{t'_m}}$  a linear combination of monomials  $(q^\sigma)^i (q^\tau)^j$  with  $i \geq k$  and  $j \geq m$ . Combined with (4) this yields the desired lower bounds on the partial degrees of the monomials  $(q^\sigma)^i (q^\tau)^j$  occurring in  $a_i^r(q^\sigma, q^\tau)$ . The explicit expression for  $a_i^r(q^\sigma, q^\tau)$  follows immediately from Proposition 4.5.9.  $\square$

As a corollary we obtain the following crucial lemma.

LEMMA 4.7.3. *Let  $-\infty < \tau < \infty$ . The limits*

$$\lim_{\sigma \rightarrow \infty} q^{r\sigma} e_r^{\sigma, \tau}, \quad \lim_{\sigma \rightarrow \infty} q^{2r\sigma} e_r^{\sigma, \sigma} \quad (1 \leq r \leq l)$$

*exist in  $A_q(U)$ . In other words, the coefficients of  $q^{r\sigma} e_r^{\sigma, \tau}$  respectively  $q^{2r\sigma} e_r^{\sigma, \sigma}$  in the expansion with respect to the monomial basis of  $A_q(U)$  tend to finite values in the limit  $\sigma \rightarrow \infty$ .*

PROOF. Fix  $1 \leq r \leq l$  and let  $-\infty < \sigma, \tau < \infty$ . From Lemma 4.7.2 it is readily deduced that

$$q^{r\sigma+r\tau} e_r = b_r^r(q^\sigma, q^\tau) \varphi_r^{\sigma, \tau} |T + \cdots + b_0^r(q^\sigma, q^\tau) \varphi_0^{\sigma, \tau} |T,$$

with  $b_i^r$  ( $0 \leq i \leq r$ ) some polynomial in two variables and  $b_r^r$  a non-zero constant polynomial (the important fact here is that  $b_i^r$  is a polynomial and not a Laurent polynomial). Hence

$$(4.7.3) \quad e_r^{\sigma, \tau} = q^{-r\sigma-r\tau} (b_r^r(q^\sigma, q^\tau) \varphi_r^{\sigma, \tau} + \cdots + b_0^r(q^\sigma, q^\tau) \varphi_0^{\sigma, \tau}).$$

Since  $\varphi_i^{\sigma, \tau} \rightarrow \varphi_i^{\infty, \tau}$  and  $\varphi_i^{\sigma, \sigma} \rightarrow \varphi_i^{\infty, \infty}$  when  $\sigma \rightarrow \infty$ , the lemma follows.  $\square$

In view of Lemma 4.7.3 we may set for  $1 \leq r \leq l$  and  $-\infty < \tau < \infty$ ,

$$(4.7.4) \quad \tilde{e}_r^{\infty, \tau} := \lim_{\sigma \rightarrow \infty} q^{r(\sigma+\tau-1)} (-1)^r e_r^{\sigma, \tau}, \quad \tilde{e}_r^{\infty, \infty} := \lim_{\sigma \rightarrow \infty} q^{r(2\sigma-1)} (-1)^r e_r^{\sigma, \sigma}.$$

It is clear from the definitions that  $\tilde{e}_r^{\infty, \tau} \in \mathcal{H}^{\infty, \tau}$  ( $1 \leq r \leq l$ ,  $-\infty < \tau \leq \infty$ ). Observe that the elements  $(-1)^r e_r^{\sigma, \tau}$  are mapped onto  $e_r(-x) \in \mathbb{C}[x^{\pm 1}]^{\mathcal{W}}$  under the restriction mapping  $|T$ . It will be shown later on that the limit transitions (4.7.4) on the level of quantum Grassmannians turn out to be the proper analogue of the limit transitions

$$(4.7.5) \quad \lim_{u \rightarrow 0} u^r e_r(u^{-1}y) = \tilde{e}_r(y), \quad (1 \leq r \leq l)$$

( $y = (y_1, \dots, y_l)$ ), where  $\tilde{e}_r(y) = \tilde{m}_{(1^r)}(y) \in \mathbb{C}[y]^{\mathfrak{S}}$  is the  $r$ th elementary symmetric polynomial ( $1 \leq r \leq l$ ). It was shown in the previous chapter that the limit transitions



(4.7.5) play a crucial role for the limits from Koornwinder polynomials to multivariable big and little  $q$ -Jacobi polynomials.

The elements  $\tilde{e}_r$  ( $1 \leq r \leq l$ ) are algebraically independent generators of the algebra  $\mathbb{C}[x]^{\mathfrak{S}}$ . In other words, we have an algebra isomorphism  $\mathbb{C}[x]^{\mathfrak{S}} \rightarrow \mathbb{C}[y]$  which will be denoted in the same way as the algebra isomorphism  $\mathbb{C}[x^{\pm 1}]^{\mathcal{W}} \rightarrow \mathbb{C}[y]$  (4.7.1):

$$(4.7.6) \quad \hat{P}(\tilde{e}_1(x), \dots, \tilde{e}_l(x)) := P(x), \quad (P \in \mathbb{C}[x]^{\mathfrak{S}}).$$

**THEOREM 4.7.4.** *Let  $-\infty < \tau < \infty$ . The elements  $\tilde{e}_r^{\infty, \tau}$  ( $1 \leq r \leq l$ ) mutually commute and are algebraically independent generators of the algebra  $\mathcal{H}^{\infty, \tau}$ . Any zonal  $(\infty, \tau)$ -spherical function  $\varphi^{\infty, \tau}(\lambda) \in \mathcal{H}^{\infty, \tau}(\lambda)$  ( $\lambda \in P_K^+$ ) is equal to a non-zero scalar multiple of*

$$\hat{P}_{\lambda^{\mathfrak{h}}}^B(\tilde{e}_1^{\infty, \tau}, \dots, \tilde{e}_l^{\infty, \tau}; 1, q^{2(n-2l)}, 1, q^{2\tau+2(n-2l)}; q^2),$$

where  $P_{\mu}^B(\cdot; a, b, c, d; t)$  is the multivariable big  $q$ -Jacobi polynomial in base  $q^2$ .

**PROOF.** The elements  $e_r^{\sigma, \tau}$  ( $1 \leq r \leq l$ ) mutually commute for  $\sigma$  finite by Corollary 4.6.3. Combined with the definition of the elements  $\tilde{e}_r^{\infty, \tau}$  (4.7.4), it follows that the  $\tilde{e}_r^{\infty, \tau}$  ( $1 \leq r \leq l$ ) mutually commute. Hence the element  $Q(\tilde{e}_1^{\infty, \tau}, \dots, \tilde{e}_l^{\infty, \tau}) \in \mathcal{H}^{\infty, \tau}$  for a polynomial  $Q \in \mathbb{C}[y]$  is well defined.

For  $\sigma$  finite and  $\lambda \in P_K^+$ , a zonal  $(\sigma, \tau)$ -spherical function  $\varphi^{\sigma, \tau}(\lambda) \in \mathcal{H}^{\sigma, \tau}(\lambda)$  is given by

$$(4.7.7) \quad \varphi^{\sigma, \tau}(\lambda) := (-q^{\sigma+\tau-1})^{|\lambda^{\mathfrak{h}}|} \hat{P}_{\lambda^{\mathfrak{h}}}(e_1^{\sigma, \tau}, \dots, e_l^{\sigma, \tau}; \underline{t}^{\sigma, \tau}; q^2)$$

with  $P_{\mu}(\cdot; \underline{t}; t)$  the Koornwinder polynomial in base  $q^2$  (cf. Theorem 4.6.5). Using the elementary properties of the Koornwinder polynomials given in Remark 2.3.11, this zonal  $(\sigma, \tau)$ -spherical function can be rewritten as

$$\varphi^{\sigma, \tau}(\lambda) = (s_{\varepsilon})^{-|\lambda^{\mathfrak{h}}|} \hat{P}_{\lambda^{\mathfrak{h}}}(s_{\varepsilon} f_1^{\varepsilon}, \dots, s_{\varepsilon}^l f_l^{\varepsilon}; \underline{t}_B(\varepsilon); q^2),$$

where  $s_{\varepsilon} := q/\varepsilon(cd)^{\frac{1}{2}}$ ,  $f_r^{\varepsilon} := (-s_{\varepsilon})^{-r} e_r^{\sigma, \tau}$ ,  $\varepsilon := q^{\sigma-(n-2l)}$  and

$$\underline{t}_B(\varepsilon) = (\varepsilon^{-1}(q^2 c/d)^{\frac{1}{2}}, -\varepsilon^{-1}(q^2 d/c)^{\frac{1}{2}}, \varepsilon a(q^2 d/c)^{\frac{1}{2}}, -\varepsilon b(q^2 c/d)^{\frac{1}{2}}),$$

with parameters  $a, b, c, d$  given by

$$(4.7.8) \quad a := 1, \quad b := q^{2(n-2l)}, \quad c := 1, \quad d := q^{2\tau+2(n-2l)}.$$

Observe that  $s_{\varepsilon} = q^{1-\sigma-\tau}$ , hence by the definition (4.7.4) of  $\tilde{e}_r^{\infty, \tau}$ ,  $\lim_{\varepsilon \downarrow 0} f_r^{\varepsilon} = \tilde{e}_r^{\infty, \tau}$  for all  $r$ . Combined with the limit transition from Koornwinder polynomials to multivariable big  $q$ -Jacobi polynomials (cf. Corollary 3.4.6), we have that  $\varphi^{\infty, \tau}(\lambda) := \lim_{\sigma \rightarrow \infty} \varphi^{\sigma, \tau}(\lambda)$  exists as limit in  $A_q(U)$ , and that

$$(4.7.9) \quad \varphi^{\infty, \tau}(\lambda) = \hat{P}_{\lambda^{\mathfrak{h}}}^B(\tilde{e}_1^{\infty, \tau}, \dots, \tilde{e}_l^{\infty, \tau}; 1, q^{2(n-2l)}, 1, q^{2\tau+2(n-2l)}; q^2)$$

with  $P_{\mu}^B(\cdot; a, b, c, d; t)$  the big  $q$ -Jacobi polynomial in base  $q^2$ . It is clear that  $\varphi^{\infty, \tau}(\lambda) \in \mathcal{H}^{\infty, \tau}(\lambda)$ , but it may be zero since the algebraic independence of the elements  $\tilde{e}_r^{\infty, \tau}$  ( $r \in [1, l]$ ) is not yet established. To prove that  $\varphi^{\infty, \tau}(\lambda)$  is non-zero, we compute the quadratic norm  $\|\varphi^{\sigma, \tau}(\lambda)\|^2$  with respect to the inner product  $\langle \phi, \psi \rangle := h(\psi^* \phi)$ , where  $h$  is the

normalized Haar functional. Since all highest weights  $\lambda \in P_K^+$  are self dual (i.e.  $V(\lambda)$  is isomorphic to its dual representation), and since the two-sided coideal  $\mathfrak{k}^\sigma$  is  $\tau$ -invariant, we have  $(\varphi^{\sigma,\tau}(\lambda))^* = \varphi^{\sigma,\tau}(\lambda)$ . Then it follows from the definition (4.7.7) of  $\varphi^{\sigma,\tau}(\lambda)$ , Corollary 4.7.1 and Theorem 2.6.6 that

$$(4.7.10) \quad \|\varphi^{\sigma,\tau}(\lambda)\|^2 = h((\varphi^{\sigma,\tau}(\lambda))^2) = s_\varepsilon^{-2|\lambda|} \frac{\mathcal{N}(\lambda^{\natural}; \underline{t}_B(\varepsilon); q^2)}{\mathcal{N}(0; \underline{t}_B(\varepsilon); q^2)},$$

where  $\mathcal{N}(\mu; \underline{t}; t)$  for  $\underline{t} \in V_K$  and  $0 < t < 1$  is the quadratic norm of the Koornwinder polynomial  $P_\mu(\cdot; \underline{t}; t)$  in base  $q^2$  with respect to the corresponding inner product  $\langle \cdot, \cdot \rangle_{\underline{t}, t}$  (cf. Theorem 2.6.6). The limit  $\varepsilon \downarrow 0$  (equivalently,  $\sigma \rightarrow \infty$ ) in (4.7.10) can now be computed in the left hand side and in the right hand side (cf. the proof of Theorem 3.4.5). It follows that

$$\|\varphi^{\infty,\tau}(\lambda)\|^2 = \frac{\mathcal{N}^B(\lambda^{\natural}; 1, q^{2(n-2l)}, 1, q^{2\tau+2(n-2l)}; q^2)}{\mathcal{N}^B(0; 1, q^{2(n-2l)}, 1, q^{2\tau+2(n-2l)}; q^2)},$$

where  $\mathcal{N}^B(\mu; a, b, c, d; t)$  is the quadratic norm of the multivariable big  $q$ -Jacobi polynomial  $P_\mu^B(\cdot; a, b, c, d; t)$  in base  $q^2$  (see Theorem 3.4.5). It follows in particular that the quadratic norm  $\|\varphi^{\infty,\tau}(\lambda)\|^2$  is non-zero, hence  $\varphi^{\infty,\tau}(\lambda) \neq 0$  for all  $\lambda \in P_K^+$ . Hence the elements  $\varphi^{\infty,\tau}(\lambda) \in \mathcal{H}^{\infty,\tau}(\lambda)$  are zonal  $(\infty, \tau)$ -spherical functions for all  $\lambda \in P_K^+$ .

It remains to prove that the  $\tilde{e}_r^{\infty,\tau}$  ( $1 \leq r \leq l$ ) are algebraically independent. Consider the finite dimensional subspaces

$$\mathcal{H}_m := \bigoplus_{\lambda \in P_K^+ : \lambda \leq m\varpi_l} \mathcal{H}^{\infty,\tau}(\lambda), \quad (m \in \mathbb{Z}_+).$$

The dimension of the linear subspace  $\mathcal{H}_m$  is equal to the number of positive integers  $\underline{m} = (m_1, \dots, m_l) \in \mathbb{Z}_+^{\times l}$  with  $|\underline{m}| := \sum_i m_i \leq m$ , since  $\varpi_r \leq \varpi_l$  for all  $r \in [0, l]$ . For such a sequence of positive integers  $\underline{m}$ , set  $Q_{\underline{m}}(y) := y_1^{m_1} \dots y_l^{m_l}$ . Since  $\tilde{e}_r^{\infty,\tau} \in \bigoplus_{s=0}^r \mathcal{H}^{\infty,\tau}(\varpi_s)$  and

$$\mathcal{H}^{\infty,\tau}(\mu) \cdot \mathcal{H}^{\infty,\tau}(\mu') \subseteq \bigoplus_{\nu \in P_K^+ : \nu \leq \mu + \mu'} \mathcal{H}^{\infty,\tau}(\nu), \quad (\mu, \mu' \in P_K^+)$$

we have  $Q_{\underline{m}}(\tilde{e}_1^{\infty,\tau}, \dots, \tilde{e}_l^{\infty,\tau}) \in \mathcal{H}_m$  for all  $\underline{m}$  with  $|\underline{m}| \leq m$ . Hence the algebraic independence of  $\tilde{e}_r^{\infty,\tau}$  ( $1 \leq r \leq l$ ) will follow from the fact that  $Q_{\underline{m}}(\tilde{e}_1^{\infty,\tau}, \dots, \tilde{e}_l^{\infty,\tau})$  ( $|\underline{m}| \leq m$ ) span  $\mathcal{H}_m$  for all  $m \in \mathbb{Z}_+$ .

Observe that  $\mathcal{H}_m$  is spanned by the zonal spherical functions  $\varphi^{\infty,\tau}(\lambda)$  ( $\lambda \leq m\varpi_l$ ). Since  $P_\mu^B(x)$  ( $\mu \in P_\Sigma^+$ ) is of the form  $\tilde{m}_\mu(x) + \sum_{\nu \in P_\Sigma^+ : \nu < \mu} c_\nu \tilde{m}_\nu(x)$  for certain constants  $c_\nu$ , it follows from the explicit expression (4.7.9) for  $\varphi^{\infty,\tau}(\lambda)$  that each  $\varphi^{\infty,\tau}(\lambda)$  with  $\lambda \leq m\varpi_l$  can be written as a linear combination of  $Q_{\underline{m}}(\tilde{e}_1^{\infty,\tau}, \dots, \tilde{e}_l^{\infty,\tau})$  ( $|\underline{m}| \leq m$ ). Hence, the monomials  $Q_{\underline{m}}(\tilde{e}_1^{\infty,\tau}, \dots, \tilde{e}_l^{\infty,\tau})$  ( $|\underline{m}| \leq m$ ) span  $\mathcal{H}_m$ .  $\square$

**THEOREM 4.7.5.** *The elements  $\tilde{e}_r^{\infty,\infty}$  ( $1 \leq r \leq l$ ) mutually commute and are algebraically independent generators of the algebra  $\mathcal{H}^{\infty,\infty}$ . Any zonal spherical function*

$\varphi^{\infty, \infty}(\lambda)$  ( $\lambda \in P_K^+$ ) is equal to a non-zero scalar multiple of

$$P_{\lambda^{\natural}}^L(\tilde{e}_1^{\infty, \infty}, \dots, \tilde{e}_l^{\infty, \infty}; q^{2(n-2l)}, 1; q^2)$$

where  $P_{\mu}^L(\cdot; a, b; t)$  is the multivariable little  $q$ -Jacobi polynomial in base  $q^2$ .

PROOF. By Theorem 4.6.5 and by symmetry properties of the Koornwinder polynomials (cf. Remark 2.3.11), a zonal  $(\sigma, \sigma)$ -spherical function  $\varphi^{\sigma, \sigma}(\lambda) \in \mathcal{H}^{\sigma, \sigma}(\lambda)$  ( $\lambda \in P_K^+$ ) is explicitly given by

$$\varphi^{\sigma, \sigma}(\lambda) := (s_{\varepsilon})^{-|\lambda^{\natural}|} \hat{P}_{\lambda^{\natural}}(s_{\varepsilon} f_1^{\varepsilon}, \dots, s_{\varepsilon}^l f_l^{\varepsilon}; \underline{t}_L(\varepsilon); q^2),$$

where  $P_{\mu}(\cdot; \underline{t}; t)$  is the Koornwinder polynomial in base  $q^2$  and  $\varepsilon := q^{2\sigma}$ ,  $s_{\varepsilon} := q/\varepsilon$ ,  $f_r^{\varepsilon} := (-s_{\varepsilon})^{-r} e_r^{\sigma, \sigma}$ , and  $\underline{t}_L(\varepsilon) := (\varepsilon^{-1}q, -aq, \varepsilon bq, -q)$  with  $a := q^{2(n-2l)}$  and  $b := 1$ . Using Corollary 3.3.4 together with the observation that  $\lim_{\varepsilon \downarrow 0} f_r^{\varepsilon} = \tilde{e}_r^{\infty, \infty}$ , the proof is analogous to the proof of Theorem 4.7.4.  $\square$

REMARK 4.7.6. Using the limits  $\tilde{e}_r^{\infty, \infty} = \lim_{\tau \rightarrow \infty} \tilde{e}_r^{\infty, \tau}$  ( $1 \leq r \leq l$ ), Theorem 4.7.5 can also be proved by sending  $\tau \rightarrow \infty$  in the results of Theorem 4.7.4. On the level of multivariable orthogonal polynomials this limit corresponds with the limit from multivariable big  $q$ -Jacobi polynomials to multivariable little  $q$ -Jacobi polynomials (3.6.25).

REMARK 4.7.7. As a corollary of Theorem 4.7.4 and Theorem 4.7.5, the restricted Haar functional  $h : \mathcal{H}^{\sigma, \tau} \rightarrow \mathbb{C}$  for  $\sigma = \infty$ ,  $-\infty < \tau < \infty$  respectively for  $\sigma = \tau = \infty$  can be expressed in terms of the orthogonality measure of the multivariable big respectively little  $q$ -Jacobi polynomials, similarly as was proved for the case  $-\infty < \sigma, \tau < \infty$  in Corollary 4.7.1.

In the last two sections we have interpreted for  $-\infty < \sigma, \tau < \infty$ ,  $\mu \in P_K^+$  the Koornwinder, the multivariable big and the multivariable little  $q$ -Jacobi polynomial

$$(4.7.11) \quad \begin{aligned} & P_{\mu}(\cdot; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+2(n-2l)+1}; q^2), \\ & P_{\mu}^B(\cdot; 1, q^{2(n-2l)}, 1, q^{2\tau+2(n-2l)}; q^2), \\ & P_{\mu}^L(\cdot; q^{2(n-2l)}, 1; q^2) \end{aligned}$$

in base  $q^2$  as zonal spherical function on quantum analogues of the complex Grassmannian. For each of these polynomials, the limit  $q \uparrow 1$  can be computed using the limit transitions (3.6.11), (3.6.9) respectively (3.6.10). The limits can be written in terms of the generalized Jacobi polynomial  $P_{\mu}^J(\cdot; n-2l, 0; 1)$  or, equivalently, in terms of the  $BC$  type Heckman-Opdam polynomial  $P_{\mu}^{HO}(\cdot; n-2l, 1, 1/2)$ . This corresponds nicely with the classical interpretation of the Heckman-Opdam polynomial  $P_{\mu}^{HO}(\cdot; n-2l, 1, 1/2)$  as zonal spherical function on the complex Grassmannian  $U/K$  (see Section 4.2).



## CHAPTER 5

# Quantum Plücker coordinates

### 5.1. Introduction

In this chapter the algebraic properties of the quantized algebra  $A_q(U/K)$  of regular functions on the complex Grassmannian

$$U/K = U(n)/(U(n-l) \times U(l)) \simeq SU(n)/S(U(n-l) \times U(l)), \quad (l \leq [n/2])$$

are studied in terms of quantum analogues of Plücker coordinates. The Plücker coordinates on the complex Grassmannian  $U/K$  were discussed in detail in Section 1.3. For the generalization to the quantum setting it is convenient to consider the Plücker coordinates from a representation theoretic viewpoint as follows.

The Dynkin diagram of  $K$  is obtained from the Dynkin diagram  $U$  by deleting its  $(n-l)$ th node. Consider the irreducible finite dimensional  $U$ -highest weight representation  $M := V((1^{n-l}))$  corresponding to the fundamental weight  $(1^{n-l})$  of the deleted node and let  $v \in M$  be a highest weight vector. Then  $K \subset U$  is the sub-group of  $U$  which stabilizes the line  $\bar{v} \in \mathbb{P}(M)$ . This induces an embedding of the homogeneous space  $U/K$  into  $\mathbb{P}(M)$ , called the Plücker embedding. Consider the linear dual  $M^*$  as functions on  $U$  by the embedding  $\phi \mapsto \phi(\cdot v)$  ( $\phi \in M^*$ ), and let  $\{\phi_i\}_i$  be a basis of  $M^*$  consisting of weight vectors. Since the weight spaces are one dimensional, the basis elements are unique up to rescaling. Then, the elements  $\phi_i(\cdot v)$  are the *holomorphic Plücker coordinates* on  $U/K$ . The holomorphic Plücker coordinates are sections of a particular line bundle over  $U/K$ . From a physical point of view, the holomorphic Plücker coordinates correspond to coherent states on the complex Grassmannian  $U/K$  (for the terminology, see Perelomov [100]). The connection of this description of the holomorphic Plücker coordinates with the description of Plücker coordinates in Section 1.3 can be made using the explicit realization of  $M$  as the  $(n-l)$ th graded piece of the exterior algebra on  $\mathbb{C}^n$ . The holomorphic Plücker coordinates correspond then to the dual Plücker coordinates of Section 1.3.

This construction of holomorphic Plücker coordinates can be immediately generalized to the quantum setting (cf. [111], [52]). Indeed, consider the irreducible finite dimensional  $U_q(\mathfrak{g})$ -representation  $M_q := V((1^{n-l}))$  of highest weight  $(1^{n-l})$ , and let  $v$  be a highest weight vector. Furthermore, let  $\{\phi_i\}_i$  be a basis of weight vectors of the linear dual  $M_q^*$ . Then, the elements  $\phi_i(\cdot v)$  are called the quantum holomorphic Plücker coordinates on  $U/K$  or, in the terminology of [52], quantum coherent states on  $U/K$ . The quantum Plücker coordinates  $\phi_i(\cdot v)$  lie in the linear dual of  $U_q(\mathfrak{g})$ . Using the doubly

non-degenerate Hopf- $*$ -algebra pairing between  $U_q(\mathfrak{g})$  and  $A_q(U)$ , they may be considered as elements in  $A_q(U)$ . Products of the form  $\phi_i(\cdot v)(\phi_j(\cdot v))^*$  lie in  $A_q(U/K)$ . The elements  $(\phi_j(\cdot v))^*$  will be called the quantum anti-holomorphic Plücker coordinates on  $U/K$ . The quantum holomorphic and anti-holomorphic Plücker coordinates can be expressed in terms of quantum minors using the realization of  $M_q$  as the  $(n-l)$ th graded piece of the quantum analogue of the exterior algebra on  $\mathbb{C}^n$ .

As was shown in the previous chapter, the modules  $W_r := \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  ( $r \in [1, l]$ ) play a crucial role in the analysis of  $A_q(U/K)$ . In fact, the irreducible decomposition of  $W_r$  is multiplicity free and the irreducible representations it contains are exactly all copies of the fundamental spherical representations  $V(\varpi_s)$  ( $s \in [0, r]$ ).

In Section 5.2 the copy of  $V(\varpi_r)$  within  $W_r$  will be realized as the kernel of an explicitly defined surjective intertwiner  $\Omega_{r,r} : W_r \rightarrow W_{r-1}$ . This intertwiner may be considered as a  $q$ -analogue of the contraction map. The contraction map  $\Omega_{r,r}$  is a type of lowering operator, in the sense that it “removes” the highest fundamental spherical part of  $W_r$ .

In the previous chapter, a surjective intertwiner  $\widehat{\Psi}_r : W_1^{\otimes r} \rightarrow W_r$  (cf. Proposition 4.5.9) was constructed. This map will be used in this chapter as a type of raising operator. Using the lowering and raising operator, it is shown in Section 5.3 that there are  $2l$  sets of elements which each generate the algebra  $A_q(U/K)$ . More precisely, for each fundamental spherical weight  $\varpi_r$  ( $r \in [1, l]$ ), two sets of generators exist which can be realized as matrix elements of the module  $W_r$ . One set of generators corresponding to the largest module  $W_l$  is exactly the set consisting of products  $\phi_i(\cdot v)(\phi_j(\cdot v))^*$  of quantum holomorphic and anti-holomorphic Plücker coordinates on  $U/K$ .

In Section 5.4 it is shown that the algebra generated by the quantum anti-holomorphic Plücker coordinates is exactly the subalgebra of  $A_q(U(n-l)) \otimes A_q(SU(l))$ -fixed elements in  $A_q(\text{Mat}(n, \mathbb{C}))$  and that the algebra generated by the quantum holomorphic and anti-holomorphic Plücker coordinates is exactly the subalgebra of  $A_q(U(n-l)) \otimes A_q(SU(l))$ -fixed elements in  $A_q(U(n))$ .

The algebraic structure of the algebra generated by the quantum anti-holomorphic Plücker coordinates was clarified by Taft and Towber [126]. In Section 5.5, their results are shortly reviewed. We replace certain arguments from [126] by representation theoretic arguments from [96].

In Section 5.6 the algebra generated by the quantum holomorphic and anti-holomorphic Plücker coordinates is studied in detail. A linear basis of the algebra in terms of “straightened” monomials of quantum Plücker coordinates is constructed, and defining relations between the quantum holomorphic and anti-holomorphic Plücker coordinates are given. The algebra  $A_q(U/K)$  can be extracted as the subalgebra of “zero-weighted” elements, i.e. it is the subalgebra spanned by straightened monomials for which the number of occurrences of quantum holomorphic Plücker coordinates is equal to the number of occurrences of quantum anti-holomorphic Plücker coordinates.

In Section 5.7, the subalgebra of bi- $A_q(K)$ -fixed elements is studied in terms of the quantum Plücker coordinates.

### 5.2. The contraction map

In the following proposition the  $q$ -analogue of the contraction map is defined. Recall the construction of the right  $A_q(U)$ -comodules  $\Lambda_q^r(V)$  and  $\Lambda_q^r(V^*)$  in Section 4.3.

PROPOSITION 5.2.1. Fix  $1 \leq r, s \leq n$ . The linear mapping

$$\Omega_{r,s} : \Lambda_q^r(V) \otimes \Lambda_q^s(V^*) \rightarrow \Lambda_q^{r-1}(V) \otimes \Lambda_q^{s-1}(V^*)$$

defined by

$$\Omega_{r,s}(v_I \otimes v_J^*) := \sum_{i \in I \cap J} \frac{s_q(i; J^c)}{s_q(I^c; i)} v_{I \setminus i} \otimes v_{J \setminus i}^* \quad (|I| = r, |J| = s)$$

is an intertwining operator of right  $A_q(U)$ -comodules.

PROOF. Denote  $R$  for the comodule mapping on  $\Lambda_q^r(V) \otimes \Lambda_q^s(V^*)$  as well as for the comodule mapping on  $\Lambda_q^{r-1}(V) \otimes \Lambda_q^{s-1}(V^*)$ . Then, on the one hand, we have

$$R \circ \Omega_{r,s}(v_I \otimes v_J^*) = \sum_{\substack{|K|=r-1 \\ |L|=s-1}} v_K \otimes v_L^* \otimes \sum_{i \in I \cap J} \frac{s_q(i; J^c)}{s_q(I^c; i)} \xi_{I \setminus i}^K (\xi_{J \setminus i}^L)^*,$$

and, on the other hand,

$$(\Omega_{r,s} \otimes \text{id}) \circ R(v_I \otimes v_J^*) = \sum_{\substack{|K|=r-1 \\ |L|=s-1}} v_K \otimes v_L^* \otimes \sum_{k \in K^c \cap L^c} \frac{s_q(k; L^c \setminus k)}{s_q(K^c \setminus k; k)} \xi_I^{K \cup k} (\xi_J^{L \cup k})^*.$$

Hence it suffices to prove that for  $|K| = r - 1$  and  $|L| = s - 1$ ,

$$(5.2.1) \quad \sum_{i \in I \cap J} \frac{s_q(i; J^c)}{s_q(I^c; i)} \xi_{I \setminus i}^K (\xi_{J \setminus i}^L)^* = \sum_{k \in K^c \cap L^c} \frac{s_q(k; L^c \setminus k)}{s_q(K^c \setminus k; k)} \xi_I^{K \cup k} (\xi_J^{L \cup k})^*.$$

Formula (5.2.1) will be proved by applying twice the Laplace expansions for quantum minors (see Section 4.3). First observe that the  $q$ -signum  $s_q$  has the following elementary properties:

$$(5.2.2) \quad \begin{aligned} s_q(I_1 \cup I_2; J) &= s_q(I_1; J) s_q(I_2; J) && \text{if } I_1 \cap I_2 = \emptyset, \\ s_q(I; J_1 \cup J_2) &= s_q(I; J_1) s_q(I; J_2) && \text{if } J_1 \cap J_2 = \emptyset, \\ s_q(I; J) s_q(J; I) &= (-q)^{|I||J|} && \text{if } I \cap J = \emptyset. \end{aligned}$$

From (5.2.2) it follows that for  $i \in I \cap J$ ,  $|I| = r$ ,  $|J| = s$ ,

$$(5.2.3) \quad \frac{s_q(J \setminus i; J^c \cup i)}{s_q(I^c; i)} = (-q)^{s-n} s_q(J; J^c) s_q(I \setminus i; i)$$

and for  $k \in K^c \cap L^c$ ,  $|K| = r - 1$ ,  $|L| = s - 1$ ,

$$(5.2.4) \quad s_q(K^c \setminus k; k)^{-1} = (-q)^{s-n} \frac{s_q(K; k) s_q(L \cup k; L^c \setminus k)}{s_q(L; L^c)}.$$

Observe that (5.2.4) follows by taking  $I := K \cup k$ ,  $J := L \cup k$  and  $i := k$  in (5.2.3). Now we compute,

$$\begin{aligned} & \sum_{i \in I \cap J} \frac{s_q(i; J^c)}{s_q(I^c; i)} \xi_{I \setminus i}^K (\xi_{J \setminus i}^L)^* = \\ &= \sum_{i \in I \cap J} \frac{s_q(J \setminus i; J^c \cup i) s_q(i; J^c)}{s_q(I^c; i) s_q(L; L^c)} \xi_{I \setminus i}^K \xi_{i \cup J^c}^{L^c} \det_q^{-1} \\ &= \sum_{k \in L^c} \frac{s_q(k; L^c \setminus k)}{s_q(L; L^c)} \left( \sum_{i \in I \cap J} \frac{s_q(J \setminus i; J^c \cup i)}{s_q(I^c; i)} \xi_{I \setminus i}^K t_{ki} \right) \xi_{J^c}^{L^c \setminus k} \det_q^{-1} \end{aligned}$$

the first equality following by (4.3.19), the second by the Laplace expansion (4.3.17),

$$= (-q)^{s-n} \frac{s_q(J; J^c)}{s_q(L; L^c)} \sum_{k \in L^c} s_q(k; L^c \setminus k) \left( \sum_{i \in I \cap J} s_q(I \setminus i; i) \xi_{I \setminus i}^K t_{ki} \right) \xi_{J^c}^{L^c \setminus k} \det_q^{-1}$$

by (5.2.3). Now, again by (4.3.17) it follows that  $\sum_{k \in L^c} s_q(k; L^c \setminus k) t_{ki} \xi_{J^c}^{L^c \setminus k} = 0$  if  $i \in J^c$ . Hence, in the last line of the above computation, the sum may be taken over  $i \in I$  instead of over  $i \in I \cap J$ . Moreover, by the Laplace expansion (4.3.18) it follows that

$$\sum_{i \in I} s_q(I \setminus i; i) \xi_{I \setminus i}^K t_{ki} = \begin{cases} s_q(K; k) \xi_I^{K \cup k} & \text{if } k \in K^c, \\ 0 & \text{if } k \in K. \end{cases}$$

Hence,

$$\begin{aligned} & \sum_{i \in I \cap J} \frac{s_q(i; J^c)}{s_q(I^c; i)} \xi_{I \setminus i}^K (\xi_{J \setminus i}^L)^* = \\ &= (-q)^{s-n} \frac{s_q(J; J^c)}{s_q(L; L^c)} \sum_{k \in K^c \cap L^c} s_q(k; L^c \setminus k) s_q(K; k) \xi_I^{K \cup k} \xi_{J^c}^{L^c \setminus k} \det_q^{-1}. \end{aligned}$$

Now (5.2.1) follows by applying (4.3.19) and (5.2.4).  $\square$

Throughout this chapter, let  $W_r$  be the right  $A_q(U)$ -comodule

$$(5.2.5) \quad W_r := \Lambda_q^r(V) \otimes \Lambda_q^r(V^*) \quad (1 \leq r \leq l).$$

By the previous proposition, we have an explicit intertwiner  $\Omega_{r,r} : W_r \rightarrow W_{r-1}$  of right  $A_q(U)$ -comodules. Our next objective is to prove that  $\Omega_{r,r}$  is surjective. To establish this result, we study the action of  $\Omega_{r,r}$  on  $A_q(K)$ -fixed vectors. In the next proposition an explicit linear basis for the space of  $A_q(K)$ -fixed vectors in  $W_r$  is given.

**PROPOSITION 5.2.2.** *Fix  $1 \leq r \leq l$ . Write*

$$\mathcal{J}_r^{(i)} := \{I \subset [1, n] \mid |I| = r, |I \cap [1, n-l]| = i\} \quad (0 \leq i \leq r),$$



and define

$$(5.2.6) \quad w_r^{(i)} := \sum_{I \in \mathfrak{J}_r^{(i)}} v_I \otimes v_I^* \in W_r.$$

Then the vectors  $w_r^{(i)}$  ( $0 \leq i \leq r$ ) form a basis for the subspace of  $A_q(K)$ -fixed vectors in  $W_r$ .

PROOF. It is obvious from the definition that the vectors  $w_r^{(i)}$  ( $0 \leq i \leq r$ ) are linearly independent. Moreover, it follows from Theorem 4.4.1 and the decomposition (4.5.14) that the subspace of  $A_q(K)$ -fixed vectors has dimension  $r+1$ . It therefore suffices to show that the elements  $w_r^{(i)}$  are  $A_q(K)$ -fixed. Let  $I, J \subset [1, n]$  be such that  $|I| = |J| = r$ . Then

$$(5.2.7) \quad \pi_K(\xi_J^I) = \begin{cases} 0 & \text{if } I \in \mathfrak{J}_r^{(i)}, J \in \mathfrak{J}_r^{(j)}, i \neq j, \\ \xi_{J_1}^{I_1} \otimes \xi_{J_2}^{I_2} & \text{if } I, J \in \mathfrak{J}_r^{(i)}, \end{cases}$$

with  $I_1 := I \cap [1, n-l]$ ,  $I_2 := \{j-n+l \mid j \in I \cap [n-l+1, n]\}$ , and similarly for  $J$ . By applying (4.3.14), (5.2.7), (4.3.19) and (4.3.16) it follows that

$$\begin{aligned} (\text{id} \otimes \pi_K)(R(w_r^{(i)})) &= \sum_{K, L \in \mathfrak{J}_r^{(i)}} v_K \otimes v_L^* \otimes \sum_{I \in \mathfrak{J}_r^{(i)}} \pi_K(\xi_I^K (\xi_I^L)^*) \\ &= \sum_{K \in \mathfrak{J}_r^{(i)}} v_K \otimes v_K^* \otimes 1 = w_r^{(i)} \otimes 1, \end{aligned}$$

which concludes the proof of the proposition.  $\square$

Observe that  $\Omega_{r,r}(w_r^{(i)})$  is an  $A_q(K)$ -fixed vector in  $W_{r-1}$  by Proposition 5.2.1 and Proposition 5.2.2, hence it can be uniquely written as a linear combination of the basis elements  $\{w_{r-1}^{(i)}\}_{i=0}^{r-1}$ . More precisely, we have the following lemma.

LEMMA 5.2.3. Fix  $1 \leq r \leq l$ . Then,  $\Omega_{r,r}(w_r^{(i)}) = c_r^{(i)} w_{r-1}^{(i-1)} + d_r^{(i)} w_{r-1}^{(i)}$  with

$$\begin{aligned} c_r^{(i)} &= \begin{cases} 0 & \text{if } i = 0, \\ (-q)^{r-n} \frac{(1-q^{2(n-l-i+1)})}{1-q^2} & \text{if } 1 \leq i \leq r, \end{cases} \\ d_r^{(i)} &= \begin{cases} (-q)^{r-n} \frac{(q^{2(n-l-i)} - q^{2(n-r+1)})}{1-q^2} & \text{if } 0 \leq i \leq r-1, \\ 0 & \text{if } i = r. \end{cases} \end{aligned}$$

PROOF. Observe that

$$\Omega_{r,r}(w_r^{(i)}) = \sum_{I \in \mathfrak{J}_{r-1}^{(i-1)}} \sum_{\substack{j \leq n-l \\ j \notin I}} \frac{s_q(j; I^c \setminus j)}{s_q(I^c \setminus j; j)} v_I \otimes v_I^* + \sum_{I \in \mathfrak{J}_{r-1}^{(i)}} \sum_{\substack{j > n-l \\ j \notin I}} \frac{s_q(j; I^c \setminus j)}{s_q(I^c \setminus j; j)} v_I \otimes v_I^*,$$

where the first term should be omitted when  $i = 0$ , and the second term should be omitted when  $i = r$ . Now

$$s_q(I^c \setminus j; j)^{-1} s_q(j; I^c \setminus j) = (-q)^{r-n} s_q(j; I^c \setminus j)^2$$

for  $I \subset [1, n]$ ,  $|I| = r - 1$ . Also, for  $I \in \mathcal{J}_{r-1}^{(i-1)}$  one has

$$\sum_{\substack{j \in [1, n-l] \\ j \notin I}} s_q(j; I^c \setminus j)^2 = \sum_{s=0}^{n-l-i} q^{2s} = \frac{1 - q^{2(n-l-i+1)}}{1 - q^2},$$

whereas for  $I \in \mathcal{J}_{r-1}^{(i)}$  one has

$$\sum_{\substack{j \in [n-l+1, n] \\ j \notin I}} s_q(j; I^c \setminus j)^2 = \sum_{s=0}^{l+i-r} q^{2(n-l-i+s)} = \frac{q^{2(n-l-i)} - q^{2(n-r+1)}}{1 - q^2}.$$

This proves the assertion.  $\square$

**COROLLARY 5.2.4.** *The intertwiner  $\Omega_{r,r}: W_r \rightarrow W_{r-1}$  is surjective. The kernel of  $\Omega_{r,r}$  is equal to the unique copy of  $V(\varpi_r)$  within  $W_r$ .*

**PROOF.** It follows from Proposition 5.2.2 and Lemma 5.2.3 that the image of  $\Omega_{r,r}$  contains the subspace of  $A_q(K)$ -fixed vectors in  $W_{r-1}$ . Since  $\Omega_{r,r}$  is an intertwiner (cf. Proposition 5.2.1), it follows from the irreducible decomposition of the comodules  $W_s$  (4.5.14) and from the fact that each  $V(\varpi_s)$  has non-zero  $A_q(K)$ -fixed vectors (cf. Theorem 4.4.1) that  $\Omega_{r,r}$  is surjective. Again by (4.5.14) and Proposition 5.2.1, it follows that the kernel of  $\Omega_{r,r}$  is the unique copy of  $V(\varpi_r)$  within  $W_r$ .  $\square$

From Lemma 5.2.3 the following explicit expression for a non-zero  $A_q(K)$ -fixed vector in  $V(\varpi_r) \subset W_r$  can be derived.

**COROLLARY 5.2.5.** *Fix  $1 \leq r \leq l$ . A non-zero  $A_q(K)$ -fixed vector in  $V(\varpi_r) \subset \Lambda_q^r(V) \otimes \Lambda_q^r(V^*)$  is given by*

$$\sum_{i=0}^r \frac{(q^{2(l-r+1)}; q^2)_i}{(q^{2(l-n)}; q^2)_i} w_r^{(i)} \in V(\varpi_r).$$

**PROOF.** The above vector is obviously non-zero,  $A_q(K)$ -fixed and an easy computation using Lemma 5.2.3 shows that the vector lies in the kernel of  $\Omega_{r,r}$ . The result follows now from Corollary 5.2.4.  $\square$

In the next proposition it is shown that the  $A_q(K)$ -fixed vectors  $w_r^{(r)}$  ( $1 \leq r \leq l$ ) can be reconstructed from  $w_1^{(1)}$  using the “raising operator”  $\widehat{\Psi}_r$  (cf. Proposition 4.5.9).

**PROPOSITION 5.2.6.** *Let  $1 \leq r \leq l$ . Then  $\widehat{\Psi}_r((w_1^{(1)})^{\otimes r}) = d_r w_r^{(r)}$  with*

$$d_r = \frac{(-q)^{\binom{r}{2}}}{(1 - q^2)^r} (q^2; q^2)_r,$$

where  $\widehat{\Psi}_r$  is the intertwiner defined in Proposition 4.5.9.

PROOF. By induction to  $r$ , the case  $r = 1$  being trivial. For the induction step, recall the intertwiner  $\widehat{\Theta}_r := (\text{pr}_{r-1,1} \otimes \text{pr}_{r-1,1}^*) \circ (\text{id}_{\Lambda_q^{r-1}(V)} \otimes \widehat{\Phi}_r \otimes \text{id}_{V^*})$  defined in Lemma 4.5.13. Then the induction step will follow from the formula

$$(5.2.8) \quad \widehat{\Theta}_r(w_{r-1}^{(r-1)} \otimes w_1^{(1)}) = (-q)^{r-1} \frac{1 - q^{2r}}{1 - q^2} w_r^{(r)}$$

for  $2 \leq r \leq l$  (compare with the proof of Proposition 4.5.9). Now it follows easily from Lemma 4.5.10 that for  $|I| = r - 1$ ,  $1 \leq i \leq n$ ,

$$\widehat{\Theta}_r(v_I \otimes v_I^* \otimes v_i \otimes v_i^*) = \begin{cases} (-q)^{r-1} v_{I \cup i} \otimes v_{I \cup i}^* & \text{if } i \notin I, \\ (-q)^{r-1} (q^2 - 1) \sum_{m < i} \left( \frac{s_q(I \setminus i; m)}{s_q(I \setminus i; i)} \right)^2 v_{I \cup m} \otimes v_{I \cup m}^* & \text{if } i \in I. \end{cases}$$

Hence

$$\widehat{\Theta}_r(w_{r-1}^{(r-1)} \otimes w_1^{(1)}) = (-q)^{r-1} \sum_{\substack{J \subset [1, n-l] \\ |J|=r}} c_J v_J \otimes v_J^*,$$

where

$$c_J := \sum_{i \in J} \left( 1 + (q^2 - 1) \sum_{\substack{j \in J \\ j > i}} \left( \frac{s_q(J \setminus (i \cup j); i)}{s_q(J \setminus (i \cup j); j)} \right)^2 \right).$$

A straightforward computation using (5.2.2) shows that

$$c_J = \sum_{s=0}^{r-1} q^{2s} = \frac{1 - q^{2r}}{1 - q^2}.$$

This proves the proposition.  $\square$

Observe that Proposition 5.2.6 can be used to prove that the ‘‘constant term’’  $a_0^r$  of  $\varphi_r^{\sigma, \tau}|_T$  (cf. Lemma 4.7.2) is non-zero. In fact,  $a_0^r$  can be computed explicitly using Proposition 5.2.6.

### 5.3. Algebraic generators of $A_q(U/K)$

Recall that the subspace of right  $A_q(K)$ -invariant functions

$$A_q(U/K) := \{\varphi \in A_q(U) \mid (\text{id} \otimes \pi_K) \circ \Delta(\varphi) = \varphi \otimes 1\},$$

is a  $*$ -subalgebra and a left  $A_q(U)$ -subcomodule of  $A_q(U)$ . The goal of this section is to give several explicit sets of generators for  $A_q(U/K)$ .

Define an inner product on  $W_r$  by

$$(5.3.1) \quad \langle v_I \otimes v_J^*, v_K \otimes v_L^* \rangle := \langle v_I, v_K \rangle \langle v_J^*, v_L^* \rangle = \delta_{I,K} \delta_{J,L} q^{-\langle 2\rho, \bar{\varepsilon}_J \rangle}$$

(cf. Section 4.3). Write  $\theta_r$  for the map of Lemma 4.6.1 associated with the unitary comodule  $(W_r, \langle \cdot, \cdot \rangle)$ . Observe that

$$(5.3.2) \quad \theta_r(v_I \otimes v_J^*, v_K \otimes v_L^*) = q^{-\langle 2\rho, \bar{\varepsilon}_J \rangle} \xi_K^I (\xi_L^J)^*,$$

where  $I, J, K, L$  are subsets of  $[1, n]$  of cardinality  $r$ .

PROPOSITION 5.3.1. *Fix  $r \in [1, l]$ . The elements*

$$\psi_r^{(i)} := \sum_{I \in \mathcal{J}_r^{(i)}} \xi_I^{[1,r]} (\xi_I^{[n-r+1,n]})^* \in A_q(U) \quad (i \in [0, r])$$

are highest weight vectors of highest weight  $\varpi_r$  with respect to the natural left  $A_q(U)$ -coaction on  $A_q(U)$ . Furthermore, the  $\psi_r^{(i)}$  ( $i \in [0, r]$ ) are right  $A_q(K)$ -invariant and they differ only by a non-zero scalar multiple.

PROOF. Write  $u_r$  for the highest weight vector  $v_{[1,r]} \otimes v_{[n-r+1,n]}^*$  of the unique copy of  $V(\varpi_r)$  within  $W_r$ . The elements  $\tilde{\psi}_r^{(i)} := \theta_r(u_r, w_r^{(i)})$  ( $i \in [0, r]$ ) are right  $A_q(K)$ -invariant and satisfy  $(\pi_- \otimes \text{id}) \circ \Delta(\tilde{\psi}_r^{(i)}) = z^{\varpi_r} \otimes \tilde{\psi}_r^{(i)}$  by Lemma 4.6.1, Proposition 5.2.2 and Remark 4.3.1. We claim that the elements  $\tilde{\psi}_r^{(i)}$  are all non-zero and that they differ only by a non-zero scalar multiple. The proof of the proposition is then completed by observing that  $\psi_r^{(i)}$  is a non-zero multiple of  $\tilde{\psi}_r^{(i)}$  for all  $i \in [0, r]$  (cf. (5.2.6), (5.3.2)).

For any  $i \in [0, r]$  let  $v_r^{(i)} \in V(\varpi_i) \subset W_r$  denote a non-zero  $A_q(K)$ -fixed vector. By Corollary 5.2.5 one has  $v_r^{(r)} = \sum_{i=0}^r \lambda_i w_r^{(i)}$  with  $\lambda_i \neq 0$  for all  $i$ . Since the inner product  $\langle \cdot, \cdot \rangle$  on  $W_r$  is invariant under the coaction of  $A_q(U)$ , the vectors  $v_r^{(i)}$  ( $0 \leq i \leq r$ ) are mutually orthogonal. We may therefore assume that the  $v_r^{(i)}$  have been rescaled such that they form an orthonormal basis of the subspace of  $A_q(K)$ -fixed vectors in  $W_r$ . Then  $w_r^{(i)} = \sum_{j=0}^r \langle w_r^{(i)}, v_r^{(j)} \rangle v_r^{(j)}$  with  $\langle w_r^{(i)}, v_r^{(r)} \rangle = \sum_{j=0}^r \lambda_j \langle w_r^{(i)}, w_r^{(j)} \rangle = \lambda_i \sum_{I \in \mathcal{J}_r^{(i)}} q^{-\langle 2\rho, \varepsilon_I \rangle} \neq 0$  for all  $0 \leq i \leq r$ . Now  $\tilde{\psi}_r^{(i)} = \langle w_r^{(i)}, v_r^{(r)} \rangle \theta_r(u_r, v_r^{(r)})$ , since the  $V(\varpi_j)$  ( $0 \leq j \leq r-1$ ) lie in the orthogonal complement of  $V(\varpi_r)$  with respect to the inner product  $\langle \cdot, \cdot \rangle$  on  $W_r$ . In particular, the  $\tilde{\psi}_r^{(i)}$  ( $0 \leq i \leq r$ ) differ only by a scalar multiple. Moreover,  $\theta_r(u_r, v_r^{(r)}) \neq 0$ , since the restriction of  $\theta_r$  to  $V(\varpi_r) \subset W_r$  is injective. It follows that  $\tilde{\psi}_r^{(i)} \neq 0$  for all  $i \in [0, r]$ .  $\square$

By the Peter-Weyl decomposition (4.3.9) and Theorem 4.4.1, the left  $A_q(U)$ -comodule  $A_q(U/K)$  has a multiplicity-free irreducible decomposition,

$$A_q(U/K) = \bigoplus_{\lambda \in P_K^+} V_L(\lambda), \quad V_L(\lambda) := W(\lambda) \cap A_q(U/K),$$

where  $V_L(\lambda)$  is irreducible of highest weight  $\lambda$ . Combined with Proposition 5.3.1 and the fact that  $A_q(U)$  has no zero divisors we obtain the following proposition.

**PROPOSITION 5.3.2.** *For  $\mu \in P_{\Sigma}^+$  a highest weight vector for the left  $A_q(U)$ -submodule  $V_L(\lambda_{\mu}) \subset A_q(U/K)$  is given by*

$$\psi_{\mu} := \psi_1^{a_1} \psi_2^{a_2} \cdots \psi_l^{a_l} \in A_q(U),$$

where  $\psi_r := \psi_r^{(r)}$  ( $1 \leq r \leq l$ ),  $a_l := \mu_l$  and  $a_i := \mu_i - \mu_{i+1} \geq 0$  ( $1 \leq i \leq l-1$ ).

Using Proposition 5.3.2 it is now possible to give several sets of algebraic generators of the subalgebra  $A_q(U/K)$ . Fix  $r \in [1, l]$ . For  $I, J \subset [1, n]$  such that  $|I| = |J| = r$  there are uniquely determined elements  $x_{IJ}^{\pm} \in A_q(U)$  such that

$$(5.3.3) \quad R(w_r^{(r)}) = \sum_{|I|=|J|=r} v_I \otimes v_J^* \otimes x_{IJ}^+, \quad R(w_r^{(0)}) = \sum_{|I|=|J|=r} v_I \otimes v_J^* \otimes x_{IJ}^-.$$

By (4.3.14), the matrix elements  $x_{IJ}^{\pm}$  are explicitly given by

$$(5.3.4) \quad x_{IJ}^+ = \sum_{\substack{K \subset [1, n-l] \\ |K|=r}} \xi_K^I (\xi_K^J)^*, \quad x_{IJ}^- = \sum_{\substack{K \subset [n-l+1, n] \\ |K|=r}} \xi_K^I (\xi_K^J)^*.$$

If  $x_{IJ}^{\varepsilon}$  is non-zero, then it is equal to  $\theta(v_I \otimes v_J^*, w_r^{(r)})$  if  $\varepsilon = +$  respectively  $\theta(v_I \otimes v_J^*, w_r^{(0)})$  if  $\varepsilon = -$  up to a non-zero constant. This implies by Lemma 4.6.1 that the  $x_{IJ}^{\varepsilon}$  are right  $A_q(K)$ -invariant.

Observe that the elements  $x_{IJ}^{\varepsilon}$  for  $r = 1$  and  $I = \{i\}, J = \{j\}$  satisfy

$$(5.3.5) \quad x_{ij}^- = \delta_{ij} - x_{ij}^+ \quad (1 \leq i, j \leq n)$$

by (4.3.16). The contraction map  $\Omega_{r,r} : W_r \rightarrow W_{r-1}$  and the intertwiner  $\widehat{\Psi}_r : (W_1)^{\otimes r} \rightarrow W_r$  can now be used as lowering operator respectively raising operator to prove the following main result of this section.

**THEOREM 5.3.3.** *Fix  $r \in [1, l]$  and  $\varepsilon \in \{+, -\}$ . Then, the elements  $x_{IJ}^{\varepsilon}$  ( $|I| = |J| = r$ ) generate the subalgebra  $A_q(U/K)$ .*

**PROOF.** For  $r \in [1, l]$  and  $\varepsilon \in \{+, -\}$ , let  $B_r^{\varepsilon} \subset A_q(U/K)$  be the unital subalgebra generated by the  $x_{IJ}^{\varepsilon}$  ( $|I| = |J| = r$ ).

Fix  $r \in [2, l]$ . By Lemma 5.2.3 we have  $\Omega_{r,r}(w_r^{(0)}) = d_r^{(0)} w_{r-1}^{(0)}$  with  $d_r^{(0)}$  a non-zero constant. Now let  $R \circ \Omega_{r,r}$  act on the vector  $w_r^{(0)}$ , where  $R$  is the right  $A_q(U)$ -coaction on  $W_{r-1}$ , and use that  $\Omega_{r,r}$  intertwines the  $A_q(U)$ -coaction, then it follows that

$$\sum_{|I|=|J|=r} \Omega_{r,r}(v_I \otimes v_J^*) \otimes x_{IJ}^- = d_r^{(0)} \sum_{|K|=|L|=r-1} v_K \otimes v_L^* \otimes x_{KL}^-.$$

Since  $v_K \otimes v_L^*$  ( $|K| = |L| = r-1$ ) form a linear basis of  $W_{r-1}$ , it follows that  $B_{r-1}^- \subseteq B_r^-$ . Similarly, it follows from  $\Omega_{r,r}(w_r^{(r)}) = c_r^{(r)} w_{r-1}^{(r-1)}$  with  $c_r^{(r)}$  a non-zero constant (cf. Lemma 5.2.3) that  $B_{r-1}^+ \subseteq B_r^+$ . Furthermore, by (5.3.5) we have  $B_1^+ = B_1^-$ . Hence we have the following inclusions of subalgebras,

$$A_q(U/K) \supseteq B_l^- \supseteq B_{l-1}^- \supseteq \cdots \supseteq B_1^- = B_1^+ \subseteq \cdots \subseteq B_{l-1}^+ \subseteq B_l^+ \subseteq A_q(U/K).$$

So it remains to prove that  $B_1^+ = A_q(U/K)$ . Since  $B_1^+$  is a left  $A_q(U)$ -comodule subalgebra of  $A_q(U/K)$ , it suffices to prove that  $\psi_r^{(r)} \in B_1^+$  for  $r \in [1, l]$  (cf. Proposition 5.3.2). Now  $\psi_r^{(r)} = x_{[1,r][n-r+1,n]}^+$ , so by the definition of  $B_1^\varepsilon$  we have  $\psi_1^{(1)} \in B_1^+$ . For  $r \in [2, l]$  we have  $\widehat{\Psi}_r((w_1^{(1)})^{\otimes r}) = d_r w_r^{(r)}$  for some non-zero constant  $d_r$  by Proposition 5.2.6. Let  $R \circ \widehat{\Psi}_r$  act on  $(w_1^{(1)})^{\otimes r}$ , where  $R$  is the right  $A_q(U)$ -coaction on  $W_r$ , and use that  $\widehat{\Psi}_r$  intertwines the  $A_q(U)$ -coactions, then it follows that

$$\sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_r}} \widehat{\Psi}_r(v_{i_1} \otimes v_{j_1}^* \otimes \cdots \otimes v_{i_r} \otimes v_{j_r}^*) \otimes x_{i_1 j_1}^+ \cdots x_{i_r j_r}^+ = d_r \sum_{|I|=|J|=r} v_I \otimes v_J^* \otimes x_{IJ}^+.$$

Since the elements  $v_I \otimes v_J^*$  ( $|I| = |J| = r$ ) form a linear basis of  $W_r$ , it follows that  $\psi_r^{(r)} = x_{[1,r][n-r+1,n]}^+ \in B_1^+$ . This concludes the proof of the theorem.  $\square$

#### 5.4. Algebras generated by quantum Plücker coordinates

To simplify the notations, we set

$$(5.4.1) \quad t_I := \xi_{[n-l+1,n]}^I = s_q(I^c; I) (\xi_{[1,n-l]}^{I^c})^* \det_q, \quad (I \subset [1, n], |I| = l).$$

In the terminology of the introduction, the elements  $(\xi_{[1,n-l]}^I)^*$  ( $|I| = n - l$ ) are the quantum anti-holomorphic Plücker coordinates. Since  $\det_q$  lies in the center of  $A_q(U)$ , the study of the algebraic relations between quantum holomorphic and anti-holomorphic quantum Plücker coordinates is equivalent with the study of the algebraic relations between the elements  $t_I, t_J^*$  ( $|I| = |J| = l$ ). This ‘‘ambiguity’’ is caused by the fact that we work here with the reductive group  $U(n)$  instead of its semisimple part  $SU(n)$  (which, in the quantum setting, amounts to dividing  $A_q(U)$  out by the relation  $\det_q = 1$ ). In the reductive setting it turns out to be more convenient to work with the elements  $t_I$  and  $t_J^*$ , and we will therefore reserve the names quantum anti-holomorphic respectively holomorphic Plücker coordinates to the elements  $t_I$  respectively  $t_J^*$ . In terms of these quantum analogues of the Plücker coordinates, the algebraic generators  $x_{IJ}^-$  of  $A_q(U/K)$  can be rewritten as  $x_{IJ}^- = t_I t_J^*$  for  $I, J \subset [1, n], |I| = |J| = l$  (cf. Theorem 5.3.3).

For the study of the algebraic properties of  $A_q(U/K)$  it is convenient to study first the algebra generated by the quantum anti-holomorphic Plücker coordinates  $t_I$  ( $|I| = l$ ) and the algebra generated by the Plücker coordinates  $t_I, t_J^*$  ( $|I| = |J| = l$ ). In this section these related algebras are characterized by certain invariance properties.

Let  $A_q(SU(n))$  be the algebra  $A_q(U(n))$  divided out by the two-sided ideal generated by  $\det_q - 1$ , and set  $\pi : A_q(U(n)) \rightarrow A_q(SU(n))$  for the natural projection. There exists a unique Hopf- $*$ -algebra structure on  $A_q(SU(n))$  such that  $\pi$  is an homomorphism of Hopf- $*$ -algebras. Any right  $A_q(U)$ -comodule  $(R, M)$  has a right  $A_q(SU(n))$ -comodule structure with comodule map given by  $(\text{id} \otimes \pi) \circ R$ . If  $M$  is irreducible as  $A_q(U)$ -comodule, then it remains irreducible as  $A_q(SU(n))$ -comodule. All irreducible

$A_q(SU(n))$ -comodules arise in this way. Furthermore,  $V(\lambda) \simeq V(\lambda')$  as  $A_q(SU(n))$ -comodule if and only if  $\lambda - \lambda' \in \mathbb{Z}(1^n)$ . Set

$$(5.4.2) \quad K_0 := U(n-l) \times SU(l), \quad A_q(K_0) := A_q(U(n-l)) \otimes A_q(SU(l))$$

and let  $\pi_K^0 : A_q(U(n)) \rightarrow A_q(K_0)$  be the surjective Hopf- $*$ -algebra homomorphism given by

$$(5.4.3) \quad \pi_K^0 := (\text{id}_{A_q(U(n-l))} \otimes \pi) \circ \pi_K.$$

Then

$$(5.4.4) \quad A_q(U/K_0) := \{\phi \in A_q(U) \mid (\text{Id} \otimes \pi_K^0) \circ \Delta(\phi) = \phi \otimes 1\}$$

is a  $*$ -subalgebra and a left  $A_q(U)$ -comodule  $*$ -subalgebra of  $A_q(U)$ .

**THEOREM 5.4.1.** *The left  $A_q(U)$ -comodule  $A_q(U/K_0)$  has the multiplicity-free irreducible decomposition*

$$A_q(U/K_0) = \bigoplus_{\lambda \in P_{K_0}^+} V_L(\lambda), \quad V_L(\lambda) := W(\lambda) \cap A_q(U/K_0),$$

where  $V_L(\lambda)$  is irreducible of highest weight  $\lambda$ . The set  $P_{K_0}^+$  of spherical weights are explicitly given by  $P_{K_0}^+ := \{\lambda_{\mu,s} \mid \mu \in P_{\Sigma}^+, s \geq -\mu_l\}$  where

$$\lambda_{\mu,s} := (\mu_1, \dots, \mu_l, 0, \dots, 0, -s - \mu_l, \dots, -s - \mu_1) \in P_n^+.$$

A highest weight vector of the left  $A_q(U)$ -subcomodule  $V_L(\lambda_{\mu,s}) \subset A_q(U/K_0)$  is given by

$$\psi_{\mu,s} := \begin{cases} \psi_{\mu} (t_{[1,l]})^{-s} & \text{if } s \leq 0, \\ \psi_{\mu} \left( (t_{[n-l+1,n]})^* \right)^s & \text{if } s \geq 0 \end{cases}$$

where  $\psi_{\mu}$  is defined in Proposition 5.3.2. Furthermore,  $A_q(U/K_0)$  is generated as algebra by the elements  $t_I, t_J^*$  ( $|I| = |J| = l$ ).

A proof of Theorem 5.4.1 can be given by repeating the arguments which were used in the previous sections and in Chapter 4 for proving the analogous results about  $A_q(U/K)$ . The details are omitted here.

We identify the bialgebra  $A_q(\text{Mat}(n, \mathbb{C}))$  with its image under the natural bialgebra embedding  $A_q(\text{Mat}(n, \mathbb{C})) \hookrightarrow A_q(U)$ . Then

$$(5.4.5) \quad A_q(\text{Mat}(n, \mathbb{C})/K_0) := \{\phi \in A_q(\text{Mat}(n, \mathbb{C})) \mid (\text{Id} \otimes \pi_K^0) \circ \Delta(\phi) = \phi \otimes 1\}$$

is a left  $A_q(U)$ -comodule subalgebra of  $A_q(\text{Mat}(n, \mathbb{C}))$ .

**COROLLARY 5.4.2.** (cf. [89, formula (17)])

The left  $A_q(U)$ -comodule  $A_q(\text{Mat}(n)/K_0)$  has the multiplicity-free irreducible decomposition

$$(5.4.6) \quad A_q(\text{Mat}(n, \mathbb{C})/K_0) = \bigoplus_{s \in \mathbb{Z}_+} V_L((s^l)),$$

where  $V_L((1^s)) := W((s^l)) \cap A_q(\text{Mat}(n, \mathbb{C})/K_0)$  is an irreducible left  $A_q(U)$ -comodule of highest weight  $(s^l)$ . A highest weight vector of the left  $A_q(U)$ -subcomodule  $V_L((s^l))$  is  $(t_{[1, l]})^s$ . Furthermore,  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$  is generated as unital algebra by the quantum Plücker coordinates  $t_I$  ( $|I| = l$ ).

PROOF. Recall that the irreducible decomposition of  $A_q(\text{Mat}(n, \mathbb{C}))$  under its natural  $A_q(U)$ -bicomodule structure is given by

$$A_q(\text{Mat}(n, \mathbb{C})) = \bigoplus_{\lambda \in P_n^+ : \lambda_n \geq 0} W(\lambda)$$

(cf. [96]). The decomposition (5.4.6) and the statement about the highest weight vector in  $V_L((s^l))$  are now direct consequences of Theorem 5.4.1. Let  $B$  be the unital subalgebra of  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$  generated by the  $t_I$  ( $|I| = l$ ). Since  $B$  contains the highest weight vectors  $(t_{[1, l]})^s$  and is stable under the left  $A_q(U)$ -coaction, it follows that  $B = A_q(\text{Mat}(n, \mathbb{C})/K_0)$ .  $\square$

COROLLARY 5.4.3. *There is a natural grading  $A_q(U/K_0) = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}_m$ , where*

$$\mathcal{A}_m := \{\phi \in A_q(U/K_0) \mid (id \otimes |T|) \circ \Delta(\phi) = \phi \otimes (z_{n-l+1} \cdots z_n)^m\}.$$

Furthermore,

$$\mathcal{A}_m = \{\phi \in A_q(U) \mid (id \otimes \pi_K) \circ \Delta(\phi) = \phi \otimes (1 \otimes \det_q^m)\}.$$

In particular,  $A_q(U/K) = \mathcal{A}_0$ .

PROOF. By direct calculations it is verified that  $t_I \in \mathcal{A}_1$  and  $t_J^* \in \mathcal{A}_{-1}$  for all subsets  $I, J \subset [1, n]$  of cardinality  $l$ . Since  $A_q(U/K_0)$  is generated as algebra by the quantum Plücker coordinates  $t_I, t_J^*$  ( $|I| = |J| = l$ ) (cf. Theorem 5.4.1), it follows that  $A_q(U/K_0) = \bigoplus_{m \in \mathbb{Z}} \mathcal{A}_m$ . The remaining assertions are straightforward.  $\square$

By Theorem 5.3.3, Theorem 5.4.1, Corollary 5.4.2 and Corollary 5.4.3 we may regard  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$  as the quantum algebra of anti-holomorphic polynomials on  $U/K$  and  $A_q(U/K_0)$  as the quantum algebra of complex-valued polynomials on  $U/K$ . Then, by Corollary 5.4.3,  $A_q(U/K)$  corresponds to the quantum algebra of zero weighted complex valued polynomials on  $U/K$ .

The subalgebra  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$  has been studied extensively by Taft and Towber [126]. In particular, in [126] an algebraic characterization is obtained for the subalgebra  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$  in terms of the quantum Plücker coordinates  $t_I$  ( $|I| = l$ ), and a linear basis is constructed in terms of straightened monomials in the  $t_I$ 's. Some of these results can be recovered from the explicit basis of the irreducible finite dimensional  $A_q(U)$ -comodules as given in [96]. Since these results will play an important role in the remainder of this chapter, they will be treated in some detail in the next section, following mainly the line of arguments from [96].



### 5.5. Algebraic properties of anti-holomorphic Plücker coordinates

In the following lemma we give explicit quadratic relations for the quantum Plücker coordinates  $t_I$ .

LEMMA 5.5.1. For  $r_1, r_2 \in [0, l]$ ,  $J_1, J_2, K \subset [1, n]$ ,  $|J_1| = l - r_1$ ,  $|J_2| = l - r_2$  and  $|K| = r_1 + r_2$ , set

$$R_q(r_1, r_2; J_1, J_2; K) := \sum_{\substack{L \subset K \\ |L|=r_2}} s_q(J_1; K \setminus L) s_q(K \setminus L; L) s_q(L; J_2) t_{J_1 \cup (K \setminus L)} t_{L \cup J_2}.$$

Then,  $R_q(r_1, r_2; J_1, J_2; K) = 0$  if  $r_1 + r_2 \geq l + 1$ .

PROOF. Follows easily from the generalized Plücker relations given in [96, Proposition 1.2] and from the definition of the quantum Plücker coordinates  $t_I$ .  $\square$

Among the relations in Lemma 5.5.1, there are two types of quadratic relations which play an important role for the algebraic structure of  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$ , namely the so-called *q-Garnir relations* and the *Young symmetry relations*. The *q-Garnir relations* are by definition the quadratic relations

$$(5.5.1) \quad R_q(l + 1 - r, r; J_1, J_2; K) = 0, \quad (r \in [1, l], |J_1| = r - 1, |J_2| = l - r, |K| = l + 1)$$

and the Young symmetry relations are by definition the quadratic relations

$$(5.5.2) \quad R_q(l, r; \emptyset, J_2; K) = 0, \quad (r \in [1, l], |J_2| = l - r, |K| = l + r).$$

It turns out that the *q-Garnir relations*, as well as the Young symmetry relations, characterize the algebraic structure of  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$  completely. Before stating the precise result, we need to introduce some more notations and terminology.

For  $\lambda \in P_n^+$  with  $\lambda_n \geq 0$ , let  $\text{SSTAB}_n(\lambda)$  be the semistandard tableaux of shape  $\lambda$  with coefficients in  $[1, n]$ . In other words,  $T \in \text{SSTAB}_n(\lambda)$  is a sequence of numbers  $T_{ij} \in [1, n]$ , where for fixed  $1 \leq i \leq n$ ,  $j$  runs through  $[1, \lambda_i]$ , such that  $T_{ij} < T_{i+1, j}$  and  $T_{ij} \leq T_{i, j+1}$  for all the relevant  $i$  and  $j$ . For  $T \in \text{SSTAB}_n(\lambda)$ , let  $T^{(j)} \subset [1, n]$  be the set of numbers in the  $j$ th column of  $T$ . Observe that the cardinality of  $T^{(j)}$  is equal to the number of boxes in the  $j$ th column of  $T$ . For  $s \in \mathbb{Z}_+$  and  $T \in \text{SSTAB}_n((s^l))$ , set

$$t_T := t_{T^{(1)}} t_{T^{(2)}} \dots t_{T^{(s)}} \in A_q(\text{Mat}(n, \mathbb{C})/K_0).$$

The elements  $t_T$  ( $T \in \text{SSTAB}_n((s^l))$ ,  $s \in \mathbb{Z}_+$ ) are called the *straightened monomials*.

The subspace  $\Lambda_q^l(V)$  spanned by the vectors  $\{v_I\}_{|I|=l}$  has a left  $A_q(U)$ -comodule structure given by the formula

$$(5.5.3) \quad L(v_I) = \sum_{|J|=l} \xi_J^l \otimes v_J \quad (|I|=l).$$

With this left  $A_q(U)$ -comodule structure,  $\Lambda_q^l(V)$  is irreducible of highest weight  $(1^l)$  with highest weight vector  $v_{[1, l]}$  and lowest weight vector  $v_{[n-l+1, n]}$ . The inner product

defined by  $\langle v_I, v_J \rangle = \delta_{I,J}$  for  $|I| = |J| = l$  is left  $A_q(U)$ -invariant, where we use the convention that a left  $A_q(U)$ -comodule  $(L, M)$  with inner product  $\langle \cdot, \cdot \rangle$  is left invariant if

$$\sum_{(m),(n)} m_{(1)}(n_{(1)})^* \langle m_{(2)}, n_{(2)} \rangle = \langle m, n \rangle 1 \quad \forall m, n \in M$$

( $\sum_{(m)} m_{(1)} \otimes m_{(2)} := L(m)$ ). Let  $T(\Lambda_q^l(V)) := \bigoplus_{s \in \mathbb{Z}_+} \Lambda_q^l(V)^{\otimes s}$  be the tensor algebra of the comodule  $\Lambda_q^l(V)$ . The tensor algebra  $T(\Lambda_q^l(V))$  has a unique left  $A_q(U)$ -comodule algebra structure such that the coaction on the tensors of degree 1 coincides with the coaction (5.5.3) on  $\Lambda_q^l(V)$ . Write  $L$  for the corresponding comodule mapping on  $T(\Lambda_q^l(V))$ . The comodule map  $L$  preserves the natural grading of the tensor algebra, i.e. the subspaces  $\Lambda_q^l(V)^{\otimes s}$  are invariant subspaces. The linear map defined by

$$\Psi_a : T(\Lambda_q^l(V)) \rightarrow A_q(\text{Mat}(n, \mathbb{C})/K_0), \quad \Psi_a(v_{I_1} \otimes v_{I_2} \otimes \dots \otimes v_{I_s}) := t_{I_1} t_{I_2} \dots t_{I_s}$$

for  $|I_j| = l$  is a surjective left  $A_q(U)$ -comodule algebra homomorphism (the subindex  $a$  of  $\Psi_a$  stands for anti-holomorphic). By Lemma 5.5.1, the elements

$$\tilde{R}_q(r_1, r_2; J_1, J_2; K) := \sum_{\substack{L \subset K \\ |L|=r_2}} s_q(J_1; K \setminus L) s_q(K \setminus L; L) s_q(L; J_2) v_{J_1 \cup (K \setminus L)} \otimes v_{L \cup J_2} \in \Lambda_q^l(V)^{\otimes 2}$$

for  $r_1, r_2 \in [0, l]$ ,  $|J_1| = l - r_2$ ,  $|J_2| = l - r_2$  and  $|K| = r_1 + r_2$  with  $r_1 + r_2 \geq l + 1$  lie in the kernel of  $\Psi_a$ . The following theorem covers the essential results from [126]. A proof of the theorem is sketched in which certain arguments from [126] are replaced by representation theoretic arguments from [96].

**THEOREM 5.5.2.** *Let  $s \in \mathbb{Z}_+$ .*

- (i)  $\Psi_a$  maps  $\Lambda_q^l(V)^{\otimes s}$  surjectively onto  $V_L((1^s)) := W((1^s)) \cap A_q(\text{Mat}(n, \mathbb{C})/K_0)$ .
- (ii) The set  $\mathcal{B}_a(s) := \{t_T \mid T \in \text{SSTAB}_n((s^l))\}$  is a linear basis of  $V_L((s^l))$ . The disjoint union  $\mathcal{B}_a := \bigcup_{s \in \mathbb{Z}_+} \mathcal{B}_a(s)$  is a linear basis of  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$ .
- (iii) The kernel of  $\Psi_a$  is generated as two-sided ideal by the  $q$ -Garnir type elements

$$\tilde{R}_q(l + 1 - r, r; J_1, J_2; K) \quad (r \in [1, l], |J_1| = r - 1, |J_2| = l - r, |K| = r + 1).$$

- (iv) The kernel of  $\Psi_a$  is generated as two-sided ideal by the Young symmetry type elements

$$\tilde{R}_q(l, r; \emptyset, J_2; K) \quad (r \in [1, l], |J_2| = l - r, |K| = l + r).$$

**PROOF.** (i) Fix arbitrary subsets  $I_j \subset [1, n]$  of cardinality  $l$  and set  $\tilde{t} := t_{I_1} t_{I_2} \dots t_{I_s}$ . We have to show that  $\tilde{t} \in W((1^s))$ .

The left  $A_q(U)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\Lambda_q^l(V)$  induces in a natural way a left  $A_q(U)$ -invariant inner product on  $\Lambda_q^l(V)^{\otimes s}$ , which will also be denoted by  $\langle \cdot, \cdot \rangle$ . Then

$$\tilde{t} = \langle L((v_{[n-l+1, n]})^{\otimes s}), v_{I_1} \otimes v_{I_2} \otimes \dots \otimes v_{I_s} \rangle,$$

where we have used the notation

$$\left\langle \sum_i a_i \otimes \phi_i, \psi \right\rangle := \sum_i a_i \langle \phi_i, \psi \rangle, \quad a_i \in A_q(U), \quad \phi_i, \psi \in \Lambda_q^l(V)^{\otimes s}.$$

Since  $(v_{[n-t+1, n]})^{\otimes s} \in \Lambda_q^l(V)^{\otimes s}$  is a lowest weight vector of the unique copy of  $V_L((s^l))$  in  $\Lambda_q^l(V)^{\otimes s}$ , it follows from the unitarity of  $\langle \cdot, \cdot \rangle$  that  $\tilde{t} \in W((s^l))$ .

(ii) In [126] it is shown that an arbitrary product  $\tilde{t} = t_{I_1} t_{I_2} \dots t_{I_s}$  of Plücker coordinates can be rewritten as linear combination of straightened monomials  $t_T$  ( $T \in \text{SSTAB}_n((s^l))$ ) by repeated application of the  $q$ -Garnir relations (5.5.1) as straightening relations. Together with (i) this implies that  $V_L((1^s))$  is spanned by the set  $\mathcal{B}_a(s)$  of straightened monomials. Since the dimension of  $V_L((s^l))$  is equal to the cardinality of  $\text{SSTAB}_n((s^l))$  (see [96]), (ii) follows.

(iii) In the proof of (ii), only the  $q$ -Garnir relations (5.5.1) are needed to rewrite an arbitrary product  $\tilde{t} = t_{I_1} t_{I_2} \dots t_{I_s}$  of Plücker coordinates as linear combination of straightened monomials  $t_T$  ( $T \in \text{SSTAB}_n((s^l))$ ). Since the straightened monomials form a basis of  $A_q(\text{Mat}(n, \mathbb{C})/K_0)$ , (iii) follows.

(iv) In [126] it was stated that the Young symmetry relations are sufficient to straighten any product of quantum Plücker coordinates, but the proof given in [126] is not complete. A complete proof of this fact can be given using [35, Lemma 6.15]. The details will be omitted here.  $\square$

REMARK 5.5.3. Theorem 5.5.2 implies in particular that the restriction of  $\Psi_a$  to the  $A_q(U)$ -invariant subspace  $\Lambda_q^l(V)$  yields a left  $A_q(U)$ -comodule isomorphism from  $\Lambda_q^l(V)$  onto  $\text{Span}\{t_I\}_{|I|=l}$  which maps  $v_I$  to  $t_I$  for all  $I$ .

The tensor algebra  $T(\Lambda_q^l(V))$  decomposes as a direct sum

$$(5.5.4) \quad T(\Lambda_q^l(V)) = \bigoplus_{s, t \in \mathbb{Z}_+} T_{s, t},$$

where  $T_{s, t}$  is the span of tensors  $\tilde{v} := v_{I_1} \otimes v_{I_2} \otimes \dots \otimes v_{I_s}$  for which the weight  $w(\tilde{v}) := \sum_{j=1}^s \sum_{i \in I_j} i$  is equal to  $t$ . This defines a grading on  $T(\Lambda_q^l(V))$ , namely

$$T_{s, t} \otimes T_{s', t'} \subset T_{s+s', t+t'}, \quad (s, s', t, t' \in \mathbb{Z}_+).$$

Observe that

$$\tilde{R}_q(r_1, r_2; J_1, J_2; K) \in T_{2, t}$$

with  $t$  the sum of the entries of  $J_1, J_2$  and  $K$ . Combined with Theorem 5.5.2 (iii), it follows that  $\text{Ker}(\Psi_a)$  is graded,

$$(5.5.5) \quad \text{Ker}(\Psi_a) = \bigoplus_{s, t \in \mathbb{Z}_+} (\text{Ker}(\Psi_a) \cap T_{s, t}).$$

On the other hand, set

$$(5.5.6) \quad A_a(s, t) := \text{Span}\{t_T \mid T \in \text{SSTAB}_n((s^l)), w(T) = t\},$$

where  $w(T) := \sum_{i, j} T_{i, j}$  is the weight of  $T$ , then we have the direct sum decomposition

$$(5.5.7) \quad A_q(\text{Mat}(n, \mathbb{C})/K_0) = \bigoplus_{s, t \in \mathbb{Z}_+} A_a(s, t)$$

by Theorem 5.5.2. Observe that  $A_a(s, t) \subseteq \Psi_a(T_{s,t})$  for all  $s, t \in \mathbb{Z}_+$ . In fact, by (5.5.5), we have the following lemma.

LEMMA 5.5.4. *For all  $s, t \in \mathbb{Z}_+$ , we have  $A_a(s, t) = \Psi_a(T_{s,t})$ . In particular, the direct sum decomposition (5.5.5) is a grading,*

$$A_a(s, t) \cdot A_a(s', t') \subset A_a(s + s', t + t'), \quad (s, s', t, t' \in \mathbb{Z}_+)$$

and for subsets  $I_j \subset [1, n]$  with  $|I_j| = l$  and  $t = \sum_{j=1}^s \sum_{i \in I_j} i$ , we have  $t_{I_1} t_{I_2} \dots t_{I_s} \in A_a(s, t)$ .

### 5.6. The algebraic structure of $A_q(U/K_0)$ and $A_q(U/K)$

In this section the algebraic structure of the algebra  $A_q(U/K_0)$  of complex-valued polynomial functions on  $U/K$  is characterized in terms of the Plücker coordinates  $t_I, t_J^*$ .

The algebraic structure of the subalgebra  $A_q^*(\text{Mat}(n)/K_0)$  of  $A_q(U)$  generated by the quantum holomorphic Plücker coordinates  $t_I^*$  ( $|I| = l$ ) follows directly from Theorem 5.5.2. Indeed, applying the  $*$ -involution to Lemma 5.5.1, quadratic relations for the Plücker coordinates  $t_I^*$  are obtained. In particular, “dual”  $q$ -Garnir relations and “dual” Young symmetry relations for the  $t_I^*$  are obtained by applying the  $*$ -involution on the relations (5.5.1) respectively (5.5.2). The map  $\Psi_a$  in Theorem 5.5.2 should be replaced by the map

$$\Psi_h : T(\Lambda_q^l(V^*)) \rightarrow A_q^*(\text{Mat}(n)/K_0), \quad \Psi_h(v_{I_1}^* \otimes \dots \otimes v_{I_s}^*) := t_{I_1}^* \dots t_{I_s}^*$$

for  $|I_j| = l$ , which is a surjective left  $A_q(U)$ -comodule algebra homomorphism if the tensor algebra  $T(\Lambda_q^l(V^*))$  is given the left  $A_q(U)$ -comodule algebra structure induced from the left coaction

$$L(v_I^*) := \sum_{|J|=l} (\xi_J^I)^* \otimes v_J^*, \quad (|I| = l)$$

on  $\Lambda_q^l(V^*)$  (here the subindex  $h$  of  $\Psi_h$  stands for holomorphic). In particular,  $\Lambda_q^l(V^*)$  is isomorphic as left  $A_q(U)$ -comodule to the linear span of the Plücker coordinates  $t_I^*$  ( $|I| = l$ ) (with comodule action given by the restriction of the comultiplication  $\Delta$  on  $A_q(U)$ ). The kernel of  $\Psi_h$  is generated as two-sided ideal by the dual  $q$ -Garnir type or, equivalently, by the dual Young symmetry type elements in  $\Lambda_q^l(V^*)^{\otimes 2}$ .

So the algebraic relations between the quantum holomorphic Plücker coordinates and the antiholomorphic Plücker coordinates have to be studied in order to clarify the algebraic structure of  $A_q(U/K_0)$ . Set

$$(5.6.1) \quad W := \Lambda_q^l(V) \oplus \Lambda_q^l(V^*),$$

and consider  $W$  as left  $A_q(U)$ -comodule. Let  $T(W)$  be the tensor algebra of  $W$  and extend the left  $A_q(U)$ -comodule structure on  $W$  to a left  $A_q(U)$ -comodule algebra structure on  $T(W)$ . The tensor algebras  $T(\Lambda_q^l(V))$  and  $T(\Lambda_q^l(V^*))$  can be naturally embedded into  $T(W)$  as left  $A_q(U)$ -comodule subalgebras. They generate  $T(W)$  as algebra.

Let  $\Psi : T(W) \rightarrow A_q(U/K_0)$  be the surjective left  $A_q(U)$ -comodule algebra homomorphism which coincides on  $T(\Lambda_q^l(V))$  (respectively  $T(\Lambda_q^l(V^*))$ ) with  $\Psi_a$  (respectively  $\Psi_b$ ). We look for elements in  $T(W)_{\leq 2} := \bigoplus_{s \leq 2} W^{\otimes s}$  which generate  $\text{Ker}(\Psi) \subset T(W)$  as a two-sided ideal.

Let  $M$  and  $N$  be two finite dimensional left  $A_q(U)$ -comodules and fix linear bases  $\{m_i\}_i$  and  $\{n_j\}_j$  for  $M$  respectively  $N$ . Let  $t_{i,j}^M, t_{s,t}^N \in A_q(U)$  be the matrix coefficients of  $M$  respectively  $N$ . The matrix coefficients are uniquely determined by the requirement that  $L(m_i) = \sum_j t_{ij}^M \otimes m_j$  and  $L(n_s) = \sum_t t_{st}^N \otimes n_t$ . Fix an intertwiner  $\Phi \in \text{Hom}_{A_q(U)}(M, N)$  and let  $c_{is} \in \mathbb{C}$  be the coefficients in the expansion  $\Phi(m_i) = \sum_s c_{is} n_s$ . Then the matrix coefficients satisfy the relations

$$\sum_j t_{ij}^M c_{jt} = \sum_s c_{is} t_{st}^N, \quad \forall i, t$$

in  $A_q(U)$ . Now recall from the proof of Theorem 5.5.2 that the  $t_I$  (respectively the  $t_J^*$ ) can be obtained as matrix coefficients of the irreducible left  $A_q(U)$ -comodule  $\Lambda_q^l(V)$  (respectively  $\Lambda_q^l(V^*)$ ). The trivial one dimensional corepresentation occurs with multiplicity one in  $W_l = \Lambda_q^l(V) \otimes \Lambda_q^l(V^*) \subset W^{\otimes 2}$  since  $\Lambda_q^l(V^*)$  is the dual representation of  $\Lambda_q^l(V)$ . Explicitly, the representation space  $U_0 \simeq \mathbb{C}$  of the trivial corepresentation is spanned by the element

$$\sum_{|I|=l} (s_q(I; I^c))^2 v_I \otimes v_I^*.$$

This follows easily using the Laplace expansions for quantum minors, see (4.3.17) and (4.3.19). So the non-zero intertwiner in  $\text{Hom}_{A_q(U)}(\mathbb{C}, \Lambda_q^l(V) \otimes \Lambda_q^l(V^*))$ , which is unique up to a non-zero constant, yields the commutation relation

$$(5.6.2) \quad \sum_{|I|=l} (s_q(I; I^c))^2 t_I t_I^* = q^{2l(n-l)}$$

in  $A_q(U/K_0)$ . Other commutation relations between the quantum holomorphic and anti-holomorphic Plücker coordinates arise from the fact that

$$(5.6.3) \quad \Lambda_q^l(V) \otimes \Lambda_q^l(V^*) \simeq \Lambda_q^l(V^*) \otimes \Lambda_q^l(V)$$

as left  $A_q(U)$ -comodules. The comodules are indeed isomorphic due to the quasi-triangular structure of the universal enveloping algebra  $U_q(\mathfrak{g})$  (see [11] for more details). Here an isomorphism (5.6.3) will be constructed using the intertwiner  $\gamma : V \otimes V \rightarrow V \otimes V$  (4.5.21) of left  $A_q(U)$ -comodules. Essentially the same techniques as in the proof of Lemma 4.5.10 can be used. The construction of the isomorphism is as follows. For  $i \in [1, n-1]$ , let  $\gamma_i \in \text{End}_{A_q(U)}(V^{\otimes n})$  be the bijective intertwiner which acts as  $\gamma$  on the  $i$ th and  $(i+1)$ th component, and as the identity on the other components. Let  $s_i \in \mathfrak{S}_n$  be the simple transposition  $s_i = (i, i+1)$ . Fix  $\sigma \in \mathfrak{S}_n$  and let  $\sigma = s_{i_1} s_{i_2} \dots s_{i_l}$  be a reduced expression for  $\sigma$ . Set

$$\gamma(\sigma) := \gamma_{i_1} \circ \gamma_{i_2} \circ \dots \circ \gamma_{i_l} \in \text{End}_{A_q(U)}(V^{\otimes n}).$$

Now  $\gamma_i \circ \gamma_j = \gamma_j \circ \gamma_i$  if  $|i - j| > 1$  and  $\gamma$  satisfies the Yang-Baxter equation (4.5.22), so  $\gamma(\sigma)$  does not depend on the choice of reduced expression by Iwahori's Theorem. Consider the element  $\chi_l \in \mathfrak{S}_n$  defined by

$$(5.6.4) \quad \chi_l := \begin{pmatrix} 1 & 2 & \dots & l & l+1 & l+2 & \dots & n \\ n-l+1 & n-l+2 & \dots & n & 1 & 2 & \dots & n-l \end{pmatrix}.$$

The length of  $\chi_l$  is equal to  $l(n-l)$ , and

$$(5.6.5) \quad \chi_l = (s_{n-l} s_{n-l+1} \dots s_{n-1}) \dots (s_2 s_3 \dots s_{l+1}) (s_1 s_2 \dots s_l)$$

is a reduced expression for  $\chi_l$ . It is not difficult to prove, using [35, Lemma 4.9], that there exists a unique bijective intertwiner

$$\tilde{\gamma} : \Lambda_q^l(V) \otimes \Lambda_q^{n-l}(V) \rightarrow \Lambda_q^{n-l}(V) \otimes \Lambda_q^l(V)$$

such that  $\tilde{\gamma} \circ (\text{pr}_l \otimes \text{pr}_{n-l}) = (\text{pr}_{n-l} \otimes \text{pr}_l) \circ \gamma(\chi_l)$ . An isomorphism (5.6.3) is now obtained using the isomorphism

$$(5.6.6) \quad \Lambda_q^l(V^*) \simeq \Lambda_q^{n-l}(V) \otimes \mathbb{C} \det_q^{-1}$$

as left  $A_q(U)$ -comodules (cf. proof of Lemma 4.5.10). To formulate the corresponding commutation relations of the matrix elements, set for subsets  $I, J, K, L \subset [1, n]$  of cardinality  $l$ ,

$$(5.6.7) \quad C_{I,J}^{K,L} := (-q)^{-l(n-l)} s_q(K^c; K) s_q(I; I^c) \langle \tilde{\gamma}(v_J \otimes v_{I^c}), v_{K^c} \otimes v_L \rangle,$$

where  $\langle \cdot, \cdot \rangle$  is the  $A_q(U)$ -invariant inner product on  $\Lambda_q^{n-l}(V) \otimes \Lambda_q^l(V)$  defined by  $\langle v_{I^c} \otimes v_J, v_{K^c} \otimes v_L \rangle = \delta_{I,K} \delta_{J,L}$ .

LEMMA 5.6.1. *The quantum holomorphic and anti-holomorphic Plücker coordinates satisfy the commutation relations*

$$(5.6.8) \quad t_L^* t_K = \sum_{|I|=|J|=l} C_{I,J}^{K,L} t_I t_J^*$$

for all subsets  $K, L \subset [1, n]$  of cardinality  $l$ .

PROOF. From the explicit expression of  $\gamma$  (4.5.21) and the explicit reduced expression (5.6.5) of  $\chi_l$  it follows that

$$\tilde{\gamma}(v_{[n-l+1, n]} \otimes v_{[1, n-l]}) = v_{[1, n-l]} \otimes v_{[n-l+1, n]}.$$

Applying the left  $A_q(U)$ -comodule action and using that  $\tilde{\gamma}$  is an intertwiner, one obtains

$$\xi_{K^c}^{[1, n-l]} \xi_L^{[n-l+1, n]} = \sum_{|I|=|J|=l} \langle \tilde{\gamma}(v_J \otimes v_{I^c}), v_{K^c} \otimes v_L \rangle \xi_J^{[n-l+1, n]} \xi_{I^c}^{[1, n-l]}$$

for  $|K| = |L| = l$ . The lemma follows now by applying the antipode  $S$  on both sides of this equation and using (4.3.19).  $\square$

For subsets  $I, J \subset [1, n]$  of equal cardinality, say  $I = \{i_1 < i_2 < \dots < i_r\}$  and  $J = \{j_1 < j_2 < \dots < j_r\}$ , write  $I \leq J$  if  $i_s \leq j_s$  for all  $s \in [1, r]$ . The following properties for the coefficients  $C_{I,J}^{K,L}$  (5.6.7) can now be derived.

LEMMA 5.6.2. *The coefficients  $C_{I,J}^{K,L}$  (5.6.7) satisfy  $C_{I,J}^{K,L} = C_{J,I}^{L,K}$ . Furthermore, the coefficients  $C_{I,J}^{K,L}$  are zero unless  $I \leq K$  and  $J \leq L$ . The diagonal coefficients are explicitly given by  $C_{I,I}^{I,I} = q^{|I \cap J^c|}$ .*

PROOF. The proof involves a straightforward induction argument using the explicit construction of  $\tilde{\gamma}$ .  $\square$

Observe that, by the results so far obtained, the following four type of elements lie in the kernel of  $\Psi$ :

$$(5.6.9) \quad \begin{aligned} & \sum_{M \subset I: |M|=r} s_q(I \setminus M; M) s_q(M; J) v_{I \setminus M} \otimes v_{M \cup J}, \\ & \sum_{M \subset I: |M|=r} s_q(I \setminus M; M) s_q(M; J) v_{M \cup J}^* \otimes v_{I \setminus M}^*, \\ & v_L^* \otimes v_K - \sum_{|S|=|T|=l} C_{S,T}^{K,L} v_S \otimes v_T^*, \\ & q^{2l(n-l)} - \sum_{|M|=l} s_q(M; M^c)^2 v_M \otimes v_M^*, \end{aligned}$$

where  $r \in [1, l]$  and  $I, J, K, L \subset [1, n]$  with  $|I| = l + r$ ,  $|J| = l - r$  and  $|K| = |L| = l$ .

THEOREM 5.6.3. **(i)** *For  $s, t \in \mathbb{Z}_+$ , let  $\mathcal{B}(s, t)$  be the set of elements  $t_S(t_T)^*$  for which the pairs  $(S, T)$  satisfy  $S \in \text{SSTAB}_n((s^l))$ ,  $T \in \text{SSTAB}_n((t^l))$  and  $S^{(s)} \neq [n - l + 1, n]$  or  $T^{(t)} \neq [n - l + 1, n]$ . Then,  $\mathcal{B} := \cup_{s, t \in \mathbb{Z}_+} \mathcal{B}(s, t)$  is a disjoint union and  $\mathcal{B}$  is a linear basis of  $A_q(U/K_0)$ .*

**(ii)** *The elements (5.6.9) generate  $\text{Ker}(\Psi)$  as a two-sided ideal.*

PROOF. **(i)**  $A_q(U/K_0)$  is spanned by the elements  $t_S(t_T)^*$  with  $S \in \text{SSTAB}_n((s^l))$ ,  $T \in \text{SSTAB}_n((t^l))$  and  $s, t \in \mathbb{Z}_+$ , in view of Theorem 5.5.2 and Lemma 5.6.1. Fix now arbitrary  $S \in \text{SSTAB}_n((s^l))$ ,  $T \in \text{SSTAB}_n((t^l))$  with  $S^{(s)} = T^{(t)} = [n - l + 1, n]$ . To show that  $\mathcal{B}$  spans  $A_q(U/K_0)$ , it suffices to prove that  $t_S(t_T)^*$  can be rewritten as linear combination of elements in  $\mathcal{B}$ . This follows inductively from the fact that  $t_S(t_T)^*$  can be rewritten as a linear combination of elements  $t_U(t_V)^*$  with  $U \in \text{SSTAB}_n((u^l))$ ,  $V \in \text{SSTAB}_n((v^l))$  where  $u \leq s$ ,  $v \leq t$  and  $w(U) \leq w(S)$ ,  $w(V) \leq w(T)$ , which in turn is a consequence of (5.6.2), Theorem 5.5.2 and Lemma 5.5.4.

The linear independence of the set  $\mathcal{B}$  can be established by comparing the leading terms of the elements in  $\mathcal{B}$  (cf. Section 4.3). The technical details, which are similar to the leading term type arguments in [96, Section 2.2], will be omitted here.

**(ii)** In the proof of **(i)** only the Young symmetry relations, dual Young symmetry relations, (5.6.2) and (5.6.8) are used to prove that  $A_q(U/K_0)$  is spanned by  $\mathcal{B}$ . Since  $\mathcal{B}$  is a basis of  $A_q(U/K_0)$ , the desired result follows.  $\square$

COROLLARY 5.6.4. *The elements  $\cup_{s \in \mathbb{Z}_+} \mathcal{B}(s, s)$  form a linear basis of  $A_q(U/K)$ .*

PROOF. The set of elements  $\cup_{s,t \in \mathbb{Z}_+ : s-t=m} \mathcal{B}(s,t)$  form a linear basis of  $\mathcal{A}_m$  (cf. Corollary 5.4.3, Theorem 5.6.3). The corollary follows now from the fact that  $\mathcal{A}_0 = A_q(U/K)$  (cf. Corollary 5.4.3).  $\square$

For rank 1, the coefficients  $C_{I,J}^{K,L}$  (5.6.7) can be computed explicitly, and we obtain the following complete list of commutation relations between the quantum Plücker coordinates  $t_{in}$  and  $t_{jn}^*$  ( $i, j \in [1, n]$ ) (cf. Theorem 5.6.3 (ii)),

$$\begin{aligned} t_{in}t_{jn} &= qt_{jn}t_{in}, & t_{jn}^*t_{in}^* &= qt_{in}^*t_{jn}^* \quad (i < j), \\ t_{in}^*t_{jn} &= qt_{jn}t_{in}^* \quad (i \neq j), \\ \sum_{i=1}^n q^{2(i-1)}t_{in}t_{in}^* &= q^{2(n-1)}, \\ t_{in}^*t_{in} &= t_{in}t_{in}^* + (1-q^2) \sum_{k < i} q^{2(k-i)}t_{kn}t_{kn}^*. \end{aligned}$$

For rank 1, Theorem 5.6.3 has been proved before, see for instance [130], [96].

### 5.7. Some remarks on the subalgebra of bi- $A_q(K)$ -fixed elements

The subalgebra of bi- $A_q(K)$ -fixed elements  $\mathcal{H}^{\infty, \infty}$  in  $A_q(U)$  was studied in the previous chapter using suitable limit arguments. In particular, we have seen that  $\mathcal{H}^{\infty, \infty}$  is a commutative algebra generated by  $l$  algebraically independent elements and that the zonal spherical functions are given in terms of multivariable little  $q$ -Jacobi polynomials (cf. Theorem 4.7.5).

For  $r \in [1, l]$ , let  $\theta_r$  be the map (4.6.2) associated with the unitary module  $(W_r, \langle \cdot, \cdot \rangle)$ , where  $\langle v_I \otimes v_J^*, v_K \otimes v_L^* \rangle = \delta_{I,K} \delta_{J,L} q^{-\langle 2\rho, \bar{\epsilon}_I \rangle}$ . Then, the elements

$$(5.7.1) \quad \Psi_r^{i,j} := \theta_r(w_r^{(i)}, w_r^{(j)}) = \sum_{I \in \mathcal{I}_r^{(i)}, J \in \mathcal{I}_r^{(j)}} q^{-\langle 2\rho, \bar{\epsilon}_I \rangle} \xi_J^I (\xi_J^I)^*, \quad i, j \in [0, r]$$

all lie in the subspace  $\oplus_{s=0}^r \mathcal{H}^{\infty, \infty}(\varpi_s) \subset \mathcal{H}^{\infty, \infty}$  (cf. Lemma 4.6.1). Recall that the zonal spherical functions  $\varphi^{\infty, \infty}(\varpi_r) \in \mathcal{H}^{\infty, \infty}(\varpi_r)$  corresponding to the fundamental spherical weights  $\varpi_r$  ( $r \in [1, l]$ ) can be realized as the matrix coefficients

$$\theta_r(v_\infty(\varpi_r), v_\infty(\varpi_r)),$$

where  $v_\infty(\varpi_r)$  is a non-zero  $A_q(K)$ -fixed vector in the unique copy of  $V(\varpi_r)$  within  $W_r$ . By Corollary 5.2.5, the following explicit expressions for the zonal spherical functions  $\varphi^{\infty, \infty}(\varpi_r) \in \mathcal{H}^{\infty, \infty}(\varpi_r)$  ( $r \in [1, l]$ ) are obtained in terms of the bi- $A_q(U/K)$ -fixed elements  $\Psi_r^{i,j}$  ( $i, j \in [0, r]$ ).

LEMMA 5.7.1. *Let  $r \in [1, l]$ . Then*

$$\sum_{i,j=0}^r \frac{(q^{2(l-r+1)}; q^2)_i (q^{2(l-r+1)}; q^2)_j}{(q^{2(l-n)}; q^2)_i (q^{2(l-n)}; q^2)_j} \Psi_r^{i,j} \in \mathcal{H}^{\infty, \infty}(\varpi_r)$$

*is a zonal spherical function.*



The bi- $A_q(K)$ -fixed elements  $\psi_i^{i,0}$  are explicitly given in terms of products of Plücker coordinates by

$$(5.7.2) \quad \Psi_i^{i,0} = \sum_{I \in \mathfrak{J}_i^{(i)}} q^{-\langle 2\rho, \varepsilon_I \rangle} t_I t_I^* \in \bigoplus_{r=0}^l \mathcal{H}^{\infty, \infty}(\varpi_r), \quad (i \in [0, l]).$$

**PROPOSITION 5.7.2.** *The elements  $\Psi_i^{i,0}$  ( $i \in [1, l]$ ) are algebraically independent generators of the algebra  $\mathcal{H}^{\infty, \infty}$  of bi- $A_q(K)$ -fixed elements in  $A_q(U)$ .*

**PROOF.** By Theorem 4.7.5 it suffices to prove that  $\{\Psi_i^{i,0}\}_{i=1}^l \cup \{1\}$  is a linear basis of the  $(l+1)$ -dimensional subspace  $\bigoplus_{r=0}^l \mathcal{H}^{\infty, \infty}(\varpi_r)$ . This is an immediate consequence of Corollary 5.6.4.  $\square$

**REMARK 5.7.3.** The algebraic independence of the elements  $\Psi_i^{i,0}$  ( $i \in [1, l]$ ) can also be proved using the monomial basis of  $A_q(U/K)$  which was constructed in the previous section (cf. Corollary 5.6.4). On the other hand, the fact that  $\mathcal{H}^{\infty, \infty}$  is a commutative algebra seems to be difficult to prove directly. The indirect proof using limit arguments involving the one-parameter family of quantum Grassmannians (cf. previous chapter) seems to be the only proof known up to this moment.



## Quantized flag manifolds and irreducible \*-representations

### 6.1. Introduction

The irreducible \*-representations of the “standard” quantization  $\mathbb{C}_q[U]$  of the algebra of functions on a compact connected simple Lie group  $U$  were classified by Soibelman [110]. He showed that a 1–1 correspondence between the equivalence classes of irreducible \*-representations of  $\mathbb{C}_q[U]$  and the symplectic leaves of the underlying Poisson bracket on  $U$  exists (cf. [109], [110]). This Poisson bracket is sometimes called Bruhat-Poisson, because its symplectic foliation is a refinement of the Bruhat decomposition of  $U$  (cf. Soibelman [109], [110]). The symplectic leaves are naturally parametrized by  $W \times T$ , where  $T \subset U$  is a maximal torus and  $W$  is the analytic Weyl group associated with  $(U, T)$ .

The 1–1 correspondence between equivalence classes of irreducible \*-representations of  $\mathbb{C}_q[U]$  and symplectic leaves of  $U$  can be formally explained by the observation that in the semi-classical limit the kernel of an irreducible \*-representation should tend to a maximal Poisson ideal. The quotient of the Poisson algebra of polynomial functions on  $U$  by this ideal is isomorphic to the Poisson algebra of functions on the symplectic leaf.

The results referred to above raise the obvious question whether the irreducible \*-representations of quantized function algebras on  $U$ -homogeneous spaces can be classified and related to the symplectic foliation of the underlying Poisson bracket. This question was already raised by Lu and Weinstein [80, Question 4.8], who studied certain Poisson brackets on  $U$ -homogeneous spaces that arise as a quotient of the Bruhat-Poisson bracket on  $U$ .

As far as I know, affirmative answers to the above mentioned question have been given so far for only three types of  $U$ -homogeneous spaces, namely Podles’s family of quantum 2-spheres [101], odd-dimensional complex quantum spheres  $SU(n+1)/SU(n)$  (cf. Vaksman and Soibelman [130]), and Stiefel manifolds  $U(n)/U(n-l)$  (cf. Podkolzin and Vainerman [102]). The relation between the irreducible \*-representations of Podles spheres and the symplectic foliation of certain covariant Poisson brackets on the 2-sphere was observed by Lu and Weinstein [81].

In this chapter, the irreducible \*-representations of a certain quantized \*-algebra of functions on a generalized flag manifold are studied. More specifically, let  $G$  denote the complexification of  $U$ . The standard parabolic subgroups  $P \subset G$  and the Bruhat-Poisson

bracket on  $G$  are defined with respect to a fixed choice of Cartan subalgebra and system of positive roots. The generalized flag manifold  $U/K$  ( $K := U \cap P$ ) naturally becomes a Poisson  $U$ -homogeneous space (cf. Lu and Weinstein [80]). The quotient Poisson bracket on  $U/K$  is also called Bruhat-Poisson in [80]. The symplectic leaves of  $U/K$  coincide with the Schubert cells of the flag manifold  $G/P \simeq U/K$ .

It is straightforward to realize a quantum analogue  $\mathbb{C}_q[K]$  of the algebra of polynomial functions on  $K$  as a quantum subgroup of  $\mathbb{C}_q[U]$ . The corresponding  $*$ -subalgebra  $\mathbb{C}_q[U/K]$  of  $\mathbb{C}_q[K]$ -invariant functions in  $\mathbb{C}_q[U]$  may be regarded as a quantization of the Poisson algebra of functions on  $U/K$  endowed with the Bruhat-Poisson bracket. The main result in this chapter is a classification of all the irreducible  $*$ -representations of  $\mathbb{C}_q[U/K]$  for an important subclass of flag manifolds, which contains in particular the irreducible compact Hermitian symmetric spaces. For this subclass it will be shown that the equivalence classes of irreducible  $*$ -representations are parametrized by the Schubert cells of  $U/K$ . It should be emphasized that the flag manifold  $U/K$  is regarded here as a real manifold. This means that the algebra of functions on  $U/K$  has a natural  $*$ -structure, which survives quantization and allows us to study  $*$ -representations in a way analogous to Soibelman's approach [110].

For an arbitrary generalized flag manifold  $U/K$  we describe in detail how irreducible  $*$ -representations of  $\mathbb{C}_q[U]$  decompose under restriction to  $\mathbb{C}_q[U/K]$ . This decomposition corresponds precisely to the way symplectic leaves in  $U$  project to Schubert cells in the flag manifold  $U/K$ . It leads immediately to a classification of the irreducible  $*$ -representations of the  $C^*$ -algebra  $C_q(U/K)$ , where  $C_q(U/K)$  is obtained by taking the closure of  $\mathbb{C}_q[U/K]$  with respect to the universal  $C^*$ -norm on  $\mathbb{C}_q[U]$ . The equivalence classes of the irreducible  $*$ -representations of  $C_q(U/K)$  are naturally parametrized by the symplectic leaves of  $U/K$  endowed with the Bruhat-Poisson bracket.

For the classification of the irreducible  $*$ -representations of the quantized function algebra  $\mathbb{C}_q[U/K]$  itself it is important to have some sort of Poincaré-Birkhoff-Witt (PBW) factorization of  $\mathbb{C}_q[U/K]$ , which in turn is closely related to the irreducible decomposition of tensor products of certain finite dimensional irreducible  $U$ -modules. Such a factorization is needed in order to develop some sort of highest weight representation theory for  $\mathbb{C}_q[U/K]$ . In Soibelman's paper [110], a crucial role is played by a similar factorization of  $\mathbb{C}_q[U]$ . From Soibelman's results a factorization of the algebra  $\mathbb{C}_q[U/T]$ , corresponding to  $P$  minimal parabolic in  $G$ , can be derived.

In this chapter a PBW type factorization for a different subclass of flag manifolds is derived using the so-called Parthasarathy-Ranga Rao-Varadarajan (PRV) conjecture. This conjecture was formulated as a follow-up to certain results in the paper [99] and was independently proved by Kumar [73] and Mathieu [88] (see also Littelmann [77]). The subclass of flag manifolds  $U/K$  considered here, can be characterized by the conditions that  $(U, K)$  is a Gelfand pair and that the Dynkin diagram of the subgroup  $K$  can be obtained from the Dynkin diagram of  $U$  by deleting one node (cf. Koornwinder [68]). These two conditions are satisfied for the irreducible compact Hermitian symmetric pairs  $(U, K)$ .

From an informal point of view, the PBW factorization in the above mentioned cases states that the quantized function algebra  $\mathbb{C}_q[U/K]$  coincides with the quantized algebra of zero-weighted complex valued polynomials on  $U/K$ . Equivalently, the PBW factorization states that  $\mathbb{C}_q[U/K]$  is generated as algebra by the product of the generalized quantum holomorphic Plücker coordinates and the generalized anti-quantum holomorphic Plücker coordinates (cf. Section 1.4). Recall that this result was already proved for the complex Grassmannian in the previous chapter (cf. Theorem 5.3.3).

The quantized algebra of zero-weighted complex valued polynomials can be naturally defined for arbitrary generalized flag manifold  $U/K$ . It is always a  $*$ -subalgebra of  $\mathbb{C}_q[U/K]$  and invariant under the  $\mathbb{C}_q[U]$ -coaction. It will be called the factorized  $*$ -algebra associated with  $U/K$ . The factorized  $*$ -algebra is closely related to the quantized algebra of holomorphic polynomials on generalized flag manifolds studied by Soibelman [111], Lakshmibai and Reshetikhin [74], [75], and Jurco and Stovicek [52] (for the classical groups), as well as to the function spaces considered by Korogodsky [70].

In this chapter the irreducible  $*$ -representations of the factorized  $*$ -algebra associated with an arbitrary flag manifold  $U/K$  are classified and it is shown that the equivalence classes of irreducible  $*$ -representations are naturally parametrized by the symplectic leaves of  $U/K$  endowed with the Bruhat-Poisson bracket. In particular, a complete classification of the irreducible  $*$ -representations of  $\mathbb{C}_q[U/K]$  is obtained whenever a PBW type factorization holds for  $\mathbb{C}_q[U/K]$ , i.e. whenever  $\mathbb{C}_q[U/K]$  is equal to its factorized  $*$ -algebra. By yet unpublished results of the author, it turns out that  $\mathbb{C}_q[U/K]$  equals its factorized  $*$ -algebra for *all* flag manifolds  $U/K$ ; combined with the results of this chapter, a complete classification of the irreducible  $*$ -representations of  $\mathbb{C}_q[U/K]$  is thus obtained.

The chapter is organized as follows. In Section 6.2 the results by Lu and Weinstein [80] and Soibelman [110] concerning the Bruhat-Poisson bracket on  $U$  and the quotient Poisson bracket on a flag manifold are reviewed. In Section 6.3 we recall some well-known results on the “standard” quantization of the universal enveloping algebra of a simple complex Lie algebra and on its finite dimensional representations. Furthermore, the construction of the corresponding quantized function algebra  $\mathbb{C}_q[U]$  is recalled and certain commutation relations between matrix coefficients of irreducible corepresentations of  $\mathbb{C}_q[U]$  are presented. They will play a crucial role in the classification of the irreducible  $*$ -representations of the factorized  $*$ -algebra. In Section 6.4 the quantized algebra  $\mathbb{C}_q[U/K]$  of functions on a flag manifold  $U/K$  and its associated factorized  $*$ -subalgebra are defined. Furthermore it is shown that the factorized  $*$ -algebra is equal to  $\mathbb{C}_q[U/K]$  for the subclass of flag manifolds referred to above. In Section 6.5 the restriction of an arbitrary irreducible  $*$ -representation of  $\mathbb{C}_q[U]$  to  $\mathbb{C}_q[U/K]$  is studied. We use here Soibelman’s explicit realization of the irreducible  $*$ -representations of  $\mathbb{C}_q[U]$  as tensor products of irreducible  $*$ -representations of  $\mathbb{C}_q[SU(2)]$  (cf. [110], see also [60], [130] for  $SU(n)$ ). As a corollary, a complete classification of the irreducible  $*$ -representations of the  $C^*$ -algebra  $C_q(U/K)$  is obtained.

Section 6.6 is devoted to the classification of the irreducible \*-representations of the factorized \*-algebra associated with an arbitrary flag manifold. The techniques used in Section 6.6 are similar to those used by Soibelman [110] for the classification of the irreducible \*-representations of  $\mathbb{C}_q[U]$ , and to those used by Joseph [49] to handle the more general problem of determining the primitive ideals of  $\mathbb{C}_q[U]$ .

## 6.2. Bruhat-Poisson brackets on flag manifolds

In this section results of Soibelman [110] and of Lu and Weinstein [80] about the Bruhat-Poisson structure on generalized flag manifolds are reviewed. Detailed discussions about the terminology related to the theory of Poisson-Lie groups are omitted. The reader who is unfamiliar with the terminology is referred to [11] and [80].

Let  $\mathfrak{g}$  be a complex simple Lie algebra with a fixed Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ . Let  $G$  be the connected simply connected Lie group with Lie algebra  $\mathfrak{g}$  (regarded here as a real analytic Lie group).

Let  $R \subset \mathfrak{h}^*$  be the root system associated with  $(\mathfrak{g}, \mathfrak{h})$  and write  $\mathfrak{g}_\alpha$  for the root space associated with  $\alpha \in R$ . Let  $\Delta = \{\alpha_1, \dots, \alpha_r\}$  be a basis of simple roots for  $R$ , and let  $R^+$  (respectively  $R^-$ ) be the set of positive (respectively negative) roots relative to  $\Delta$ . We identify  $\mathfrak{h}$  with its dual by the Killing form  $\kappa$ . The non degenerate symmetric bilinear form on  $\mathfrak{h}^*$  induced by  $\kappa$  is denoted by  $(\cdot, \cdot)$ . Let  $W \subset GL(\mathfrak{h}^*)$  be the Weyl group of the root system  $R$  and write  $s_i = s_{\alpha_i}$  for the simple reflection associated with  $\alpha_i \in \Delta$ .

For  $\alpha \in R$  write  $d_\alpha := (\alpha, \alpha)/2$ . Let  $H_\alpha \in \mathfrak{h}$  be the element associated with the coroot  $\alpha^\vee := d_\alpha^{-1}\alpha \in \mathfrak{h}^*$  under the identification  $\mathfrak{h} \simeq \mathfrak{h}^*$ . Choose nonzero  $X_\alpha \in \mathfrak{g}_\alpha$  ( $\alpha \in R$ ) such that for all  $\alpha, \beta \in R$  we have  $[X_\alpha, X_{-\alpha}] = H_\alpha$ ,  $\kappa(X_\alpha, X_{-\alpha}) = d_\alpha^{-1}$  and  $[X_\alpha, X_\beta] = c_{\alpha, \beta} X_{\alpha+\beta}$  with  $c_{\alpha, \beta} = -c_{-\alpha, -\beta} \in \mathbb{R}$  whenever  $\alpha + \beta \in R$ . Let  $\mathfrak{h}_0$  be the real form of  $\mathfrak{h}$  defined as the real span of the  $H_\alpha$ 's ( $\alpha \in R$ ). Then

$$(6.2.1) \quad \mathfrak{u} := \sum_{\alpha \in R^+} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \sum_{\alpha \in R^+} \mathbb{R}i(X_\alpha + X_{-\alpha}) \oplus i\mathfrak{h}_0$$

is a compact real form of  $\mathfrak{g}$ .

Set  $\mathfrak{b} := \mathfrak{h}_0 \oplus \mathfrak{n}_+$  with  $\mathfrak{n}_+ := \sum_{\alpha \in R^+} \mathfrak{g}_\alpha$ . Then, by the Iwasawa decomposition for  $\mathfrak{g}$ , the triple  $(\mathfrak{g}, \mathfrak{u}, \mathfrak{b})$  is a Manin triple with respect to the imaginary part of the Killing form  $\kappa$  (cf. [80, Section 4]). Hence  $\mathfrak{u}$ ,  $\mathfrak{b}$  and  $\mathfrak{g}$  naturally become Lie bialgebras. The dual Lie algebra  $\mathfrak{u}^*$  is isomorphic to  $\mathfrak{b}$ , and  $\mathfrak{g}$  may be identified with the classical double of  $\mathfrak{u}$ . The cocommutator  $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$  of the Lie bialgebra  $\mathfrak{g}$  is coboundary, i.e.,

$$\delta(X) = (\text{ad}_X \otimes 1 + 1 \otimes \text{ad}_X)r,$$

with the classical  $r$ -matrix  $r \in \mathfrak{g} \wedge \mathfrak{g}$  given by the following well-known skew solution of the Modified Classical Yang-Baxter Equation,

$$(6.2.2) \quad r = i \sum_{\alpha \in R^+} d_\alpha (X_{-\alpha} \otimes X_\alpha - X_\alpha \otimes X_{-\alpha}) \in \mathfrak{u} \wedge \mathfrak{u}.$$

The cocommutator on  $\mathfrak{u}$  coincides with the restriction of  $\delta$  to  $\mathfrak{u}$ .

The corresponding Sklyanin bracket on the connected subgroup  $U \subset G$  with Lie algebra  $\mathfrak{u}$  has

$$(6.2.3) \quad \Omega_g = l_g^{\otimes 2} r - r_g^{\otimes 2} r, \quad (g \in U)$$

as its associated Poisson tensor. Here  $l_g$  respectively  $r_g$  denote infinitesimal left respectively right translation. This particular Sklyanin bracket is often called Bruhat-Poisson, since its symplectic foliation is closely related to the Bruhat decomposition of  $G$ , as will be shown in a moment.

Let  $B$  be the connected subgroup of  $G$  with Lie algebra  $\mathfrak{b}$ , let  $T \subset U$  be the maximal torus in  $U$  with Lie algebra  $i\mathfrak{h}_0$ , and set  $B_+ := TB$ . The analytic Weyl group  $N_U(T)/T$ , where  $N_U(T)$  is the normalizer of  $T$  in  $U$ , is isomorphic to  $W$ . More explicitly, the isomorphism sends the simple reflection  $s_i$  to  $\exp(\frac{\pi}{2}(X_{\alpha_i} - X_{-\alpha_i}))/T$ . The double  $B_+$ -cosets in  $G$  are parametrized by the elements of  $W$ . Hence we have the Bruhat decomposition

$$G = \coprod_{w \in W} B_+ w B_+.$$

By [133, Proposition 1.2.3.6] the Bruhat decomposition has the refinement

$$(6.2.4) \quad G = \coprod_{m \in N_U(T)} BmB.$$

For  $m \in N_U(T)$  we set  $\Sigma_m := U \cap BmB$ . Then  $\Sigma_m \neq \emptyset$  for all  $m \in N_U(T)$ , and we have the disjoint union

$$(6.2.5) \quad U = \coprod_{m \in N_U(T)} \Sigma_m.$$

Now recall that multiplication  $U \times B \rightarrow G$  is a global diffeomorphism by the Iwasawa decomposition of  $G$ . So for any  $b \in B$  and  $u \in U$  there exists a unique  $u^b \in U$  such that  $bu \in u^b B$ . As is easily verified, the map

$$(6.2.6) \quad U \times B \rightarrow U, \quad (u, b) \mapsto u^{b^{-1}}$$

is a right action of  $B$  on  $U$ , and the corresponding decomposition of  $U$  into  $B$ -orbits coincides with the decomposition (6.2.5). On the other hand, if we regard  $B$  as the Poisson-Lie group dual to  $U$ , the action (6.2.6) becomes the right dressing action of the dual group on  $U$  (cf. [80, Theorem 3.14]). Since the orbits in  $U$  under the right dressing action are exactly the symplectic leaves of the Poisson bracket on  $U$  (cf. [108, Theorem 13], [80, Theorem 3.15]), it follows that (6.2.5) coincides with the decomposition of  $U$  into symplectic leaves (cf. [110, Theorem 2.2]).

Next, we recall some results by Lu and Weinstein [80] concerning certain quotient Poisson brackets on generalized flag manifolds. Let  $S \subset \Delta$  be a set of simple roots, and let  $P_S$  be the corresponding standard parabolic subgroup of  $G$ . The Lie algebra  $\mathfrak{p}_S$  of  $P_S$

is given by

$$(6.2.7) \quad \mathfrak{p}_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma_S} \mathfrak{g}_\alpha$$

with  $\Gamma_S := R^+ \cup \{\alpha \in R \mid \alpha \in \text{span}(S)\}$ . Let  $\mathfrak{l}_S$  be the Levi factor of  $\mathfrak{p}_S$ ,

$$(6.2.8) \quad \mathfrak{l}_S := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma_S \cap (-\Gamma_S)} \mathfrak{g}_\alpha,$$

and set  $\mathfrak{k}_S := \mathfrak{p}_S \cap \mathfrak{u} = \mathfrak{l}_S \cap \mathfrak{u}$ . Then  $\mathfrak{k}_S$  is a compact real form of  $\mathfrak{l}_S$ . Set  $K_S := U \cap P_S \subset U$ , then  $K_S \subset U$  is a Poisson-Lie subgroup of  $U$  with Lie algebra  $\mathfrak{k}_S$  (cf. [80, Theorem 4.7]). Hence there is a unique Poisson bracket on  $U/K_S$  such that the natural projection  $\pi : U \rightarrow U/K_S$  is a Poisson map. This bracket is also called Bruhat-Poisson. It is covariant in the sense that the natural left action  $U \times U/K_S \rightarrow U/K_S$  is a Poisson map.

Let  $W_S$  be the subgroup of  $W$  generated by the simple reflections in  $S$ , then we have  $P_S = B_+ W_S B_+$  (cf. [133, Theorem 1.2.1.1]). It follows that the double cosets  $B_+ x P_S$  ( $x \in G$ ) are parametrized by the elements of  $W/W_S$ . Hence we have the Schubert cell decomposition of  $U/K_S \simeq G/P_S$ :

$$(6.2.9) \quad U/K_S = \coprod_{\bar{w} \in W/W_S} X_{\bar{w}}, \quad X_{\bar{w}} := (U \cap B_+ w P_S)/K_S \simeq B_+ w/P_S,$$

where  $\bar{w} \in W/W_S$  is the right  $W_S$ -coset in  $W$  which contains  $w$ .

Now, by [80, Proposition 4.5], the subgroup  $K_S$  is invariant under the action of  $B$ , which implies that the  $B$ -action descends to  $U/K_S$ . The orbits in  $U/K_S$  coincide exactly with the Schubert cells. By [80, Theorem 4.6] the symplectic leaves of the Poisson manifold  $U/K_S$  are exactly the orbits under the  $B$ -action. We conclude (cf. [80, Theorem 4.7]):

**THEOREM 6.2.1.** *The Schubert cells of the flag manifold  $U/K_S$  are the symplectic leaves of  $U/K_S$  endowed with the Bruhat-Poisson bracket.*

Consider now the set of minimal coset representatives

$$(6.2.10) \quad W^S := \{w \in W \mid l(ws_\alpha) > l(w) \quad \forall \alpha \in S\}.$$

$W^S$  is a complete set of coset representatives for  $W/W_S$ , i.e. any element  $w \in W$  can be uniquely written as a product  $w = w_1 w_2$  with  $w_1 \in W^S$ ,  $w_2 \in W_S$ . The elements of  $W^S$  are minimal in the sense that

$$(6.2.11) \quad l(w_1 w_2) = l(w_1) + l(w_2), \quad (w_1 \in W^S, w_2 \in W_S),$$

where  $l(w) := \#(R^+ \cap wR^-)$  is the length function on  $W$ .

Observe that  $\pi$  maps the symplectic leaf  $\Sigma_m \subset U$  onto the symplectic leaf  $X_{\overline{w(m)}} \subset U/K_S$ , where  $w(m) := m/T \in W$ . We write  $\pi_m : \Sigma_m \rightarrow X_{\overline{w(m)}}$  for the surjective Poisson map obtained by restricting  $\pi$  to the symplectic leaf  $\Sigma_m$ . The minimality condition (6.2.10) translates to the following property of the map  $\pi_m$ .



PROPOSITION 6.2.2. *Let  $m \in N_U(T)$ . Then  $\pi_m : \Sigma_m \rightarrow X_{\overline{w(m)}}$  is a symplectic automorphism if and only if  $w(m) \in W^S$ .*

PROOF. For  $w \in W$  set

$$\mathfrak{n}_w := \bigoplus_{\alpha \in R^+ \cap wR^-} \mathfrak{g}_\alpha, \quad N_w := \exp(\mathfrak{n}_w).$$

Observe that the complex dimension of  $N_w$  is equal to  $l(w)$ . Write  $\mathrm{pr}_U : G \simeq U \times B \rightarrow U$  for the canonical projection. It is well-known that for  $m \in N_U(T)$  and for  $w \in W^S$  with representative  $m_w \in N_U(T)$ , the maps

$$\begin{aligned} \phi_m : N_{w(m)} &\rightarrow \Sigma_m, & n &\mapsto \mathrm{pr}_U(nm), \\ \psi_w : N_w &\rightarrow X_{\overline{w}}, & n &\mapsto \pi(\mathrm{pr}_U(nm_w)) \end{aligned}$$

are surjective diffeomorphisms (see for example [9, Proposition 1.1 and 5.1]). The map  $\psi_w$  is independent of the choice of representative  $m_w$  for  $w$ . It follows now from (6.2.11) by a dimension count that  $\pi_m$  can only be a diffeomorphism if  $w(m) \in W^S$ . On the other hand, if  $m \in N_U(T)$  such that  $w(m) \in W^S$ , then  $\pi_m = \psi_{w(m)} \circ \phi_m^{-1}$  and hence  $\pi_m$  is a diffeomorphism.  $\square$

Soibelman [110] gave a description of the symplectic leaves  $\Sigma_m$  ( $m \in N_U(T)$ ) as a product of two-dimensional leaves which turns out to have a nice generalization to the quantized setting (cf. Section 6.5). For  $i \in [1, r]$ , let  $\gamma_i : SU(2) \hookrightarrow U$  be the embedding corresponding to the  $i$ th node of the Dynkin diagram of  $U$ . After a possible renormalization of the Bruhat-Poisson structure on  $SU(2)$ ,  $\gamma_i$  becomes an embedding of Poisson-Lie groups. Recall that the two-dimensional leaves of  $SU(2)$  are given by

$$S_t := \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2) \mid \arg(\beta) = \arg(t) \right\} \quad (t \in \mathbb{T})$$

where  $\mathbb{T} \subset \mathbb{C}$  is the unit circle in the complex plane. The restriction of the embedding  $\gamma_i$  to  $S_1 \subset SU(2)$  is a symplectic automorphism from  $S_1$  onto the symplectic leaf  $\Sigma_{m_i} \subset U$ , where  $m_i = \exp(\frac{\pi}{2}(X_{\alpha_i} - X_{-\alpha_i}))$ . Recall that  $m_i \in N_U(T)$  is a representative of the simple reflection  $s_i \in W$ .

For  $m \in N_U(T)$  let  $w(m) = s_{i_1} s_{i_2} \cdots s_{i_l}$  be a reduced expression for  $w(m) := m/T \in W$ , and let  $t_m \in T$  be the unique element such that  $m = m_{i_1} m_{i_2} \cdots m_{i_l} t_m$ . Note that  $t_m$  depends on the choice of reduced expression for  $w(m)$ . The map

$$(g_1, \dots, g_l) \mapsto \gamma_{i_1}(g_1) \gamma_{i_2}(g_2) \cdots \gamma_{i_l}(g_l) t_m$$

defines a symplectic automorphism from  $S_1^{\times l}$  onto the symplectic leaf  $\Sigma_m \subset U$  (cf. [110, Section 2], [114]). Observe that the image of this map is independent of the choice of reduced expression for  $w(m)$ , although the map itself is not.

Combined with Proposition 6.2.2 we now obtain the following description of the symplectic leaves of the generalized flag manifold  $U/K_S$ .

**PROPOSITION 6.2.3.** *Let  $m \in N_U(T)$  and set  $w := m/T \in W$ . Let  $w_1 \in W^S$ ,  $w_2 \in W_S$  be such that  $w = w_1 w_2$  and choose reduced expressions  $w_1 = s_{i_1} \cdots s_{i_p}$  and  $w_2 = s_{i_{p+1}} \cdots s_{i_l}$ . Then the map*

$$(g_1, g_2, \dots, g_l) \mapsto \gamma_{i_1}(g_1) \gamma_{i_2}(g_2) \cdots \gamma_{i_l}(g_l) / K_S$$

*is a surjective Poisson map from  $S_1^{\times l}$  onto the Schubert cell  $X_{\bar{w}}$ . It factorizes through the projection  $pr: S_1^{\times l} = S_1^{\times p} \times S_1^{\times(l-p)} \rightarrow S_1^{\times p}$ . The quotient map from  $S_1^{\times p}$  onto  $X_{\bar{w}}$  is a symplectic automorphism. In particular, we have*

$$X_{\bar{w}} = (\Sigma_{m_{i_1}} \Sigma_{m_{i_2}} \cdots \Sigma_{m_{i_p}}) / K_S. \quad \square$$

See Lu [79] for more details in the case of the full flag manifold ( $K_S = T$ ).

### 6.3. Preliminaries on the quantized function algebra $\mathbb{C}_q[U]$

In this section some notations are introduced which are needed throughout the remainder of this chapter. First, we recall the definition of the quantized universal enveloping algebra associated with the simple complex Lie algebra  $\mathfrak{g}$ . We use the notations introduced in the previous section.

Set  $d_i := d_{\alpha_i}$  and  $H_i := H_{\alpha_i}$  for  $i \in [1, r]$ . Let  $A = (a_{ij})$  be the Cartan matrix, i.e.  $a_{ij} := d_i^{-1}(\alpha_i, \alpha_j)$ . Note that  $H_i \in \mathfrak{h}$  is the unique element such that  $\alpha_j(H_i) = a_{ij}$  for all  $j$ . The weight lattice is given by

$$(6.3.1) \quad P = \{\lambda \in \mathfrak{h}^* \mid \lambda(H_i) = (\lambda, \alpha_i^\vee) \in \mathbb{Z} \quad \forall i\}.$$

The fundamental weights  $\varpi_{\alpha_i} = \varpi_i$  ( $i \in [1, r]$ ) are characterized by  $\varpi_i(H_j) = \delta_{ij}$  for all  $j$ . The set of dominant weights  $P_+$  respectively regular dominant weights  $P_{++}$  is equal to  $\mathbb{K}\text{-span}\{\varpi_\alpha\}_{\alpha \in \Delta}$  with  $\mathbb{K} = \mathbb{Z}_+$  respectively  $\mathbb{N}$ .

The quantized universal enveloping algebra  $U_q(\mathfrak{g})$  associated with the simple Lie algebra  $\mathfrak{g}$  is the unital associative algebra over  $\mathbb{C}$  with generators  $K_i^{\pm 1}$ ,  $X_i^\pm$  ( $i = [1, r]$ ) and relations

$$(6.3.2) \quad \begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1 \\ K_i X_j^\pm K_i^{-1} &= q_i^{\pm \alpha_j(H_i)} X_j^\pm \\ X_i^+ X_j^- - X_j^- X_i^+ &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \binom{1-a_{ij}}{s}_{q_i} (X_i^\pm)^{1-a_{ij}-s} X_j^\pm (X_i^\pm)^s &= 0 \quad (i \neq j) \end{aligned}$$

where  $q_i := q^{d_i}$ ,

$$[a]_q := \frac{q^a - q^{-a}}{q - q^{-1}} \quad (a \in \mathbb{N}), \quad [0]_q := 1,$$

$[a]_q! := [a]_q[a-1]_q \dots [1]_q$ , and

$$\binom{a}{n}_q := \frac{[a]_q!}{[a-n]_q![n]_q!}.$$

A Hopf algebra structure on  $U_q(\mathfrak{g})$  is uniquely determined by the formulas

$$(6.3.3) \quad \begin{aligned} \Delta(X_i^+) &= X_i^+ \otimes 1 + K_i \otimes X_i^+, & \Delta(X_i^-) &= X_i^- \otimes K_i^{-1} + 1 \otimes X_i^-, \\ \Delta(K_i^{\pm 1}) &= K_i^{\pm 1} \otimes K_i^{\pm 1}, \\ S(K_i^{\mp 1}) &= K_i^{\mp 1}, & S(X_i^+) &= -K_i^{-1} X_i^+, & S(X_i^-) &= -X_i^- K_i, \\ \varepsilon(K_i^{\pm 1}) &= 1, & \varepsilon(X_i^{\pm}) &= 0. \end{aligned}$$

In fact,  $U_q(\mathfrak{g})$  may be regarded as a quantization of the co-Poisson-Hopf algebra structure (cf. [11, Ch. 6]) on  $U(\mathfrak{g})$  induced by the Lie bialgebra  $(\mathfrak{g}, -i\delta)$ ,  $\delta$  being the cocommutator of  $\mathfrak{g}$  associated with the  $r$ -matrix (6.2.2).  $U_q(\mathfrak{g})$  becomes a Hopf  $*$ -algebra with  $*$ -structure on the generators given by

$$(6.3.4) \quad (K_i^{\pm 1})^* = K_i^{\pm 1}, \quad (X_i^+)^* = q_i^{-1} X_i^- K_i, \quad (X_i^-)^* = q_i K_i^{-1} X_i^+.$$

In the classical limit  $q \rightarrow 1$ , the  $*$ -structure becomes an involutive, conjugate-linear anti-automorphism of  $\mathfrak{g}$  with  $-1$  eigenspace equal to the compact real form  $\mathfrak{u}$  defined in (6.2.1).

Let  $U^{\pm} = U_q(\mathfrak{n}_{\pm})$  be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $X_i^{\pm}$  ( $i = [1, r]$ ) and write  $U^0 := U_q(\mathfrak{h})$  for the commutative subalgebra generated by  $K_i^{\pm 1}$  ( $i = [1, r]$ ). Let  $Q$  (respectively  $Q^+$ ) be the integral (respectively positive integral) span of the positive roots. We have the direct sum decomposition

$$U^{\pm} = \bigoplus_{\alpha \in Q^+} U_{\pm\alpha}^{\pm},$$

where  $U_{\pm\alpha}^{\pm} := \{\phi \in U^{\pm} \mid K_i \phi K_i^{-1} = q_i^{\pm\alpha(H_i)} \phi\}$ . The Poincaré-Birkhoff-Witt Theorem for  $U_q(\mathfrak{g})$  states that multiplication defines an isomorphism of vector spaces

$$U^- \otimes U^0 \otimes U^+ \rightarrow U_q(\mathfrak{g}).$$

In particular,  $U_q(\mathfrak{g})$  is spanned by elements of the form  $b_{-\eta} K^{\alpha} a_{\zeta}$  where  $b_{-\eta} \in U_{-\eta}^-$ ,  $a_{\zeta} \in U_{\zeta}^+$  ( $\eta, \zeta \in Q_+$ ) and  $\alpha \in Q$ . Here we used the notation  $K^{\alpha} = K_1^{k_1} \dots K_r^{k_r}$  if  $\alpha = \sum_i k_i \alpha_i$ .

For a left  $U_q(\mathfrak{g})$ -module  $V$ , we say that  $0 \neq v \in V$  has weight  $\mu \in \mathfrak{h}^*$  if  $K_i \cdot v = q_i^{\mu(H_i)} v = q^{(\mu, \alpha_i)} v$  for all  $i$ . We write  $V_{\mu}$  for the corresponding weight space. Recall that a  $P$ -weighted finite dimensional irreducible representation of  $U_q(\mathfrak{g})$  is a highest weight module  $V = V(\lambda)$  with highest weight  $\lambda \in P_+$ . If  $v_{\lambda} \in V(\lambda)$  is a highest weight vector, we have  $V(\lambda) = \sum_{\alpha \in Q^+}^{\oplus} U_{-\alpha}^- v_{\lambda}$  by the PBW Theorem, hence the set of weights  $P(\lambda)$  of  $V(\lambda)$  is a subset of the weight lattice  $P$  satisfying  $\mu \leq \lambda$  for all  $\mu \in P(\lambda)$ . Here  $\leq$  is the dominance order on  $P$  (i.e.  $\mu \leq \nu$  if  $\nu - \mu \in Q^+$  and  $\mu < \nu$  if  $\mu \leq \nu$  and  $\mu \neq \nu$ ).

We define irreducible finite dimensional  $P$ -weighted right  $U_q(\mathfrak{g})$ -modules with respect to the opposite Borel subgroup. So the irreducible finite dimensional right  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  with highest weight  $\lambda \in P^+$  has the weight space decomposition  $V(\lambda) = \sum_{\alpha \in Q^+}^{\oplus} v_\alpha U_\alpha^+$ , where  $v_\alpha \in V(\lambda)$  is the highest weight vector of  $V(\lambda)$ . The weights of the right  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  coincide with the weights of the left  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  and the dimensions of the corresponding weight spaces are the same.

The quantized algebra  $\mathbb{C}_q[G]$  of functions on the connected simply connected complex Lie group  $G$  with Lie algebra  $\mathfrak{g}$  is the subspace in the linear dual  $U_q(\mathfrak{g})^*$  spanned by the matrix coefficients of the finite dimensional irreducible representations  $V(\lambda)$  ( $\lambda \in P_+$ ). The Hopf \*-algebra structure on  $U_q(\mathfrak{g})$  induces a Hopf \*-algebra structure on the quantized function algebra  $\mathbb{C}_q[G] \subset U_q(\mathfrak{g})^*$  by the formulas

$$(6.3.5) \quad \begin{aligned} (\phi\psi)(X) &= (\phi \otimes \psi)\Delta(X), & 1(X) &= \varepsilon(X) \\ \Delta(\phi)(X \otimes Y) &= \phi(XY), & \varepsilon(\phi) &= \phi(1) \\ S(\phi)(X) &= \phi(S(X)), & (\phi^*)(X) &= \overline{\phi(S(X)^*)}, \end{aligned}$$

where  $\phi, \psi \in \mathbb{C}_q[G] \subset U_q(\mathfrak{g})^*$  and  $X, Y \in U_q(\mathfrak{g})$ . The algebra  $\mathbb{C}_q[G]$  can be regarded as a quantization of the Poisson algebra of polynomial functions on the algebraic Poisson-Lie group  $G$ , where the Poisson structure on  $G$  is given by the Sklyanin bracket associated with the classical  $r$ -matrix  $-ir$  (cf. (6.2.2)). Since the \*-structure (6.3.5) on  $\mathbb{C}_q[G]$  is associated with the compact real form  $U$  of  $G$  in the classical limit, we will write  $\mathbb{C}_q[U]$  for  $\mathbb{C}_q[G]$  with this particular choice of \*-structure. Note that  $\mathbb{C}_q[U]$  is a  $U_q(\mathfrak{g})$ -bimodule with the left respectively right action given by

$$(6.3.6) \quad (X.\phi)(Y) := \phi(YX), \quad (\phi.X)(Y) := \phi(XY)$$

where  $\phi \in \mathbb{C}_q[U]$  and  $X, Y \in U_q(\mathfrak{g})$ . The finite dimensional irreducible  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  of highest weight  $\lambda \in P_+$  is unitarizable. Write  $(\cdot, \cdot)$  for a unitary inner product on  $V(\lambda)$ . Choose an orthonormal basis of  $V(\lambda)$  with respect to  $(\cdot, \cdot)$ ,

$$(6.3.7) \quad \{v_\mu^{(i)} \mid \mu \in P(\lambda), i = [1, \dim(V(\lambda)_\mu)]\},$$

consisting of weight vectors  $v_\mu^{(i)} \in V(\lambda)_\mu$ . The index  $i$  will be omitted if  $\dim(V(\lambda)_\mu) = 1$ . Set

$$(6.3.8) \quad C_{\mu, i, \nu, j}^\lambda(X) := (X.v_\nu^{(j)}, v_\mu^{(i)}), \quad X \in U_q(\mathfrak{g}),$$

for  $\mu, \nu \in P(\lambda)$  and  $1 \leq i \leq \dim(V(\lambda)_\mu)$ ,  $1 \leq j \leq \dim(V(\lambda)_\nu)$ . If  $\dim(V(\lambda)_\mu) = 1$  respectively  $\dim(V(\lambda)_\nu) = 1$  then the dependence on  $i$  respectively  $j$  in (6.3.8) is omitted. It is sometimes also convenient to use the notation

$$C_{v; w}^\lambda(X) := (X.w, v), \quad v, w \in V(\lambda), \quad X \in U_q(\mathfrak{g}).$$

Note that when  $\lambda$  runs through  $P_+$  and  $\mu, i, \nu$  and  $j$  run through the above-mentioned sets the matrix elements (6.3.8) form a linear basis of  $\mathbb{C}_q[G]$ . Furthermore, we have the

formulas

$$(6.3.9) \quad \begin{aligned} \Delta(C_{\mu,i;\nu,j}^\lambda) &= \sum_{\sigma,s} C_{\mu,i;\sigma,s}^\lambda \otimes C_{\sigma,s;\nu,j}^\lambda, \\ \varepsilon(C_{\mu,i;\nu,j}^\lambda) &= \delta_{\mu,\nu} \delta_{i,j}, \quad (C_{\mu,i;\nu,j}^\lambda)^* = S(C_{\nu,j;\mu,i}^\lambda), \end{aligned}$$

where sums for which the summation sets are not specified are taken over the ‘‘obvious’’ choice of summation sets. Using the relations (6.3.9) and the Hopf algebra axiom for the antipode  $S$  we obtain

$$(6.3.10) \quad \sum_{\sigma,s} (C_{\sigma,s;\mu,i}^\lambda)^* C_{\sigma,s;\nu,j}^\lambda = \delta_{\mu,\nu} \delta_{i,j}.$$

The elements  $(C_{\mu,i;\nu,j}^\lambda)^*$  are matrix coefficients of the dual representation  $V(\lambda)^* \simeq V(-\sigma_0\lambda)$  (here  $\sigma_0$  is the longest element in  $W$ ). In fact, let  $\pi : U_q(\mathfrak{g}) \rightarrow \text{End}(V(\lambda))$  be the representation of highest weight  $\lambda$ , and let  $(\cdot, \cdot)$  be an inner product with respect to which  $\pi$  is unitarizable. Fix an orthonormal basis of weight vectors  $\{v_\mu^{(r)}\}$ . Write  $\pi^*$  for the dual representation, i.e.  $\pi^*(X)\phi = \phi \circ \pi(S(X))$  for  $X \in U_q(\mathfrak{g})$  and  $\phi \in V(\lambda)^*$ . For  $u \in V(\lambda)$  set  $u^* := (\cdot, u) \in V(\lambda)^*$ . Define an inner product on  $V(\lambda)^*$  by

$$(u^*, v^*) := (\pi(K^{-2\rho})v, u), \quad u, v \in V(\lambda),$$

where  $\rho = 1/2 \sum_{\alpha \in R^+} \alpha \in \mathfrak{h}^*$ . Since  $S^2(u) = K^{-2\rho}uK^{2\rho}$  for  $u \in U_q(\mathfrak{g})$  and  $(* \circ S)^2 = \text{Id}$  on  $U_q(\mathfrak{g})$ , it follows that  $\pi^*$  is unitarizable with respect to the inner product  $(\cdot, \cdot)$  on  $V(\lambda)^*$  and that  $\{\phi_{-\mu}^{(i)} := q^{(\mu,\rho)}(v_\mu^{(i)})^*\}$  is an orthonormal basis of  $V(\lambda)^*$  consisting of weight vectors (here  $\phi_{-\mu}^{(i)}$  has weight  $-\mu$ ). Defining the matrix coefficients  $C_{-\mu,i;-\nu,j}^{-\sigma_0\lambda}$  of  $(\pi^*, V(\lambda)^*)$  with respect to the orthonormal basis  $\{\phi_{-\mu}^{(i)}\}$ , we then have

$$(6.3.11) \quad (C_{\mu,i;\nu,j}^\lambda)^* = q^{(\mu-\nu,\rho)} C_{-\mu,i;-\nu,j}^{-\sigma_0\lambda}$$

(cf. [110, Proposition 3.3]). A fundamental role in Soibelman’s theory of irreducible  $*$ -representations of  $\mathbb{C}_q[U]$  is played by a Poincaré-Birkhoff-Witt (PBW) type factorization of  $\mathbb{C}_q[U]$ . For  $\lambda \in P_+$ , set

$$(6.3.12) \quad B_\lambda := \text{span}\{C_{v;v_\lambda}^\lambda \mid v \in V(\lambda)\}.$$

Note that  $B_\lambda$  is a right  $U_q(\mathfrak{g})$ -submodule of  $\mathbb{C}_q[U]$  isomorphic to  $V(\lambda)$ . Set

$$(6.3.13) \quad A^+ := \bigoplus_{\lambda \in P_+} B_\lambda, \quad A^{++} := \bigoplus_{\lambda \in P_{++}} B_\lambda.$$

The subalgebra and right  $U_q(\mathfrak{g})$ -module  $A^+$  is equal to the subalgebra of left  $U^+$ -invariant elements in  $\mathbb{C}_q[U]$  (cf. [49]). The existence of a PBW type factorization of  $\mathbb{C}_q[U]$  now amounts to the following statement.

**THEOREM 6.3.1.** [110, Theorem 3.1] *The multiplication map  $m : (A^{++})^* \otimes A^{++} \rightarrow \mathbb{C}_q[U]$  is surjective.*

A detailed proof can be found in [49, Proposition 9.2.2]. The proof is based on certain results concerning decompositions of tensor products of irreducible finite dimensional  $U_q(\mathfrak{g})$ -modules which can be traced back to Kostant in the classical case [71, Theorem 5.1]. The close connection between Theorem 6.3.1 and the decomposition of tensor products of irreducible  $U_q(\mathfrak{g})$ -modules becomes clear by observing that

$$(6.3.14) \quad (B_\lambda)^* B_\mu \simeq V(\lambda)^* \otimes V(\mu)$$

as right  $U_q(\mathfrak{g})$ -modules.

Important for the study of \*-representations of  $\mathbb{C}_q[U]$  is some detailed information about the commutation relations between matrix elements in  $\mathbb{C}_q[U]$ . In view of Theorem 6.3.1, we are especially interested in commutation relations between the  $C_{\mu,i;\lambda}^\lambda$  and  $C_{\nu,j;\Lambda}^\Lambda$  respectively between the  $C_{\mu,i;\lambda}^\lambda$  and  $(C_{\nu,j;\Lambda}^\Lambda)^*$ , where  $\lambda, \Lambda \in P_+$ . To state these commutation relations we need to introduce certain vector subspaces of  $\mathbb{C}_q[U]$ . Let  $\lambda, \Lambda \in P_+$  and  $\mu \in P(\lambda), \nu \in P(\Lambda)$ , then we set

$$(6.3.15) \quad \begin{aligned} N(\mu, \lambda; \nu, \Lambda) &:= \text{span}\{C_{v;v_\lambda}^\lambda C_{w;v_\Lambda}^\Lambda \mid (v, w) \in sN\}, \\ N^{opp}(\mu, \lambda; \nu, \Lambda) &:= \text{span}\{C_{w;v_\Lambda}^\Lambda C_{v;v_\lambda}^\lambda \mid (v, w) \in sN\} \end{aligned}$$

where  $sN := sN(\mu, \lambda; \nu, \Lambda)$  is the set of pairs  $(v, w) \in V(\lambda)_{\mu'} \times V(\Lambda)_{\nu'}$  with  $\mu' > \mu$ ,  $\nu' < \nu$  and  $\mu' + \nu' = \mu + \nu$ . Furthermore, set

$$(6.3.16) \quad \begin{aligned} O(\mu, \lambda; \nu, \Lambda) &:= \text{span}\{(C_{v;v_\lambda}^\lambda)^* C_{w;v_\Lambda}^\Lambda \mid (v, w) \in sO\}, \\ O^{opp}(\mu, \lambda; \nu, \Lambda) &:= \text{span}\{C_{w;v_\Lambda}^\Lambda (C_{v;v_\lambda}^\lambda)^* \mid (v, w) \in sO\} \end{aligned}$$

where  $sO := sO(\mu, \lambda; \nu, \Lambda)$  is the set of pairs  $(v, w) \in V(\lambda)_{\mu'} \times V(\Lambda)_{\nu'}$  with  $\mu' < \mu$ ,  $\nu' < \nu$  and  $\mu - \mu' = \nu - \nu'$ . If  $sN$  (respectively  $sO$ ) is empty, then let  $N = N^{opp} = \{0\}$  (respectively  $O = O^{opp} = \{0\}$ ).

**PROPOSITION 6.3.2.** *Let  $\lambda, \Lambda \in P_+$  and  $v \in V(\lambda)_\mu, w \in V(\Lambda)_\nu$ .*

(i) *The matrix elements  $C_{v;v_\lambda}^\lambda$  and  $C_{w;v_\Lambda}^\Lambda$  satisfy the commutation relation*

$$C_{v;v_\lambda}^\lambda C_{w;v_\Lambda}^\Lambda = q^{(\lambda, \Lambda) - (\mu, \nu)} C_{w;v_\Lambda}^\Lambda C_{v;v_\lambda}^\lambda \text{ mod } N(\mu, \lambda; \nu, \Lambda).$$

*Moreover, we have  $N = N^{opp}$ .*

(ii) *The matrix elements  $(C_{v;v_\lambda}^\lambda)^*$  and  $C_{w;v_\Lambda}^\Lambda$  satisfy the commutation relation*

$$(C_{v;v_\lambda}^\lambda)^* C_{w;v_\Lambda}^\Lambda = q^{(\mu, \nu) - (\lambda, \Lambda)} C_{w;v_\Lambda}^\Lambda (C_{v;v_\lambda}^\lambda)^* \text{ mod } O(\mu, \lambda; \nu, \Lambda).$$

*Moreover, we have  $O = O^{opp}$ .*

Soibelman [110] derived commutation relations using the universal  $R$ -matrix whereas Joseph [49, Section 9.1] used the Poincaré-Birkhoff-Witt Theorem for  $U_q(\mathfrak{g})$  and the left respectively right action (6.3.6) of  $U_q(\mathfrak{g})$  on  $\mathbb{C}_q[U]$ . Although the commutation relations formulated here are slightly sharper, the proof can be derived in a similar manner and will therefore be omitted.

COROLLARY 6.3.3. *Let  $\lambda, \Lambda \in P_+$  and  $v \in V(\lambda)_\mu$ ,  $w \in V(\Lambda)_\nu$ . Then*

$$(6.3.17) \quad C_{v;v_\lambda}^\lambda C_{w;v_\Lambda}^\Lambda = q^{(\mu,\nu) - (\lambda,\Lambda)} C_{w;v_\Lambda}^\Lambda C_{v;v_\lambda}^\lambda \pmod{N(\nu, \Lambda; \mu, \lambda)}.$$

Note that Proposition 6.3.2(i) and Corollary 6.3.3 give two different ways to rewrite  $C_{v;v_\lambda}^\lambda C_{w;v_\Lambda}^\Lambda$  as elements of the vector space

$$W_{\lambda,\Lambda} := \text{span}\{C_{w';v_\Lambda}^\Lambda C_{v';v_\lambda}^\lambda \mid v' \in V(\lambda), w' \in V(\Lambda)\}.$$

We will need both “inequivalent” commutation relations (Proposition 6.3.2(i) and Corollary 6.3.3) in later sections. It follows in particular that, when  $v' \in V(\lambda)$  and  $w' \in V(\Lambda)$  run through a basis, the elements  $C_{w';v_\Lambda}^\Lambda C_{v';v_\lambda}^\lambda$  are (in general) linearly dependent. This also follows from the following two observations. On the one hand,  $W_{\lambda,\Lambda} \simeq V(\lambda + \Lambda)$  as right  $U_q(\mathfrak{g})$ -modules. On the other hand,  $V(\lambda + \Lambda)$  occurs with multiplicity one in  $V(\lambda) \otimes V(\Lambda)$ , whereas in general  $V(\lambda) \otimes V(\Lambda)$  has other irreducible components too.

By contrast, the commutation relation given in Proposition 6.3.2(ii) is unique in the sense that, when  $v \in V(\lambda)$  and  $w \in V(\Lambda)$  run through a basis, the  $C_{w;v_\Lambda}^\Lambda (C_{v;v_\lambda}^\lambda)^*$  are linearly independent (cf. (6.3.14)).

We end this section by recalling the special case  $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ . Set

$$(6.3.18) \quad \begin{aligned} t_{11} &:= C_{\varpi_1; \varpi_1}^{\varpi_1}, & t_{12} &:= C_{\varpi_1; -\varpi_1}^{\varpi_1}, \\ t_{21} &:= C_{-\varpi_1; \varpi_1}^{\varpi_1}, & t_{22} &:= C_{-\varpi_1; -\varpi_1}^{\varpi_1}. \end{aligned}$$

Then it is well-known that the  $t_{ij}$ 's generate the algebra  $\mathbb{C}_q[SU(2)]$ . The commutation relations

$$(6.3.19) \quad \begin{aligned} t_{k1}t_{k2} &= qt_{k2}t_{k1}, & t_{1k}t_{2k} &= qt_{2k}t_{1k} \quad (k = 1, 2), \\ t_{12}t_{21} &= t_{21}t_{12}, & t_{11}t_{22} - t_{22}t_{11} &= (q - q^{-1})t_{12}t_{21}, \\ t_{11}t_{22} - qt_{12}t_{21} &= 1 \end{aligned}$$

characterize the algebra structure of  $\mathbb{C}_q[SU(2)]$  in terms of the generators  $t_{ij}$ . The  $*$ -structure is uniquely determined by the formulas  $t_{11}^* = t_{22}$ ,  $t_{12}^* = -qt_{21}$ .

#### 6.4. Quantized function algebras on generalized flag manifolds

Let  $S$  be any subset of the simple roots  $\Delta$ . Sometimes  $S$  will be identified with the index set  $\{i \mid \alpha_i \in S\}$ . Let  $\mathfrak{p}_S \subset \mathfrak{g}$  be the corresponding standard parabolic subalgebra, given explicitly by (6.2.7). Define the quantized universal enveloping algebra  $U_q(\mathfrak{l}_S)$  associated with the Levi factor  $\mathfrak{l}_S$  of  $\mathfrak{p}_S$  as the subalgebra of  $U_q(\mathfrak{g})$  generated by  $K_i^{\pm 1}$  ( $i \in [1, r]$ ) and  $X_i^\pm$  ( $i \in S$ ). Observe that  $U_q(\mathfrak{l}_S)$  is a Hopf  $*$ -subalgebra of  $U_q(\mathfrak{g})$ .

For later use in this section we briefly discuss the finite dimensional representation theory of  $U_q(\mathfrak{l}_S)$ . Recall that  $\mathfrak{l}_S$  is a reductive Lie algebra with centre

$$(6.4.1) \quad Z(\mathfrak{l}_S) = \bigcap_{i \in S} \text{Ker}(\alpha_i) \subset \mathfrak{h}.$$

Moreover, we have direct sum decompositions

$$(6.4.2) \quad \mathfrak{h} = Z(\mathfrak{l}_S) \oplus \mathfrak{h}_S, \quad \mathfrak{l}_S = Z(\mathfrak{l}_S) \oplus \mathfrak{l}_S^0,$$

where  $\mathfrak{h}_S = \text{span}\{H_i\}_{i \in S}$  and  $\mathfrak{l}_S^0$  is the semisimple part of  $\mathfrak{l}_S$ . The semisimple part  $\mathfrak{l}_S^0$  is explicitly given by

$$(6.4.3) \quad \mathfrak{l}_S^0 := \mathfrak{h}_S \oplus \bigoplus_{\alpha \in \Gamma_S \cap (-\Gamma_S)} \mathfrak{g}_\alpha.$$

Define the quantized universal enveloping algebra  $U_q(\mathfrak{l}_S^0)$  associated with the semisimple part  $\mathfrak{l}_S^0$  of  $\mathfrak{l}_S$  as the subalgebra of  $U_q(\mathfrak{g})$  generated by  $K_i^{\pm 1}$  and  $X_i^\pm$  for all  $i \in S$ . Observe that  $U_q(\mathfrak{l}_S^0)$  is a Hopf \*-subalgebra of  $U_q(\mathfrak{g})$ .

**PROPOSITION 6.4.1.** *Any finite dimensional  $U_q(\mathfrak{l}_S)$ -module  $V$  which is completely reducible as  $U_q(\mathfrak{h})$ -module, is completely reducible as  $U_q(\mathfrak{l}_S)$ -module.*

**PROOF.** Let  $V$  be a finite dimensional left  $U_q(\mathfrak{l}_S)$ -module which is completely reducible as  $U_q(\mathfrak{h})$ -module. Then the linear subspace

$$V^+ := \{v \in V \mid X_i^+ v = 0 \quad \forall i \in S\}$$

is  $U_q(\mathfrak{h})$ -stable and splits as a direct sum of weight spaces. Let  $\{v_i\}_i$  be a linear basis of  $V^+$  consisting of weight vectors, and set  $V_i := U_q(\mathfrak{l}_S^0)v_i$ . Since  $U_q(\mathfrak{l}_S^0)$  is the quantized universal enveloping algebra associated with a semisimple Lie algebra, it follows that  $V = \sum_i^\oplus V_i$  is a decomposition of  $V$  into irreducible  $U_q(\mathfrak{l}_S^0)$ -modules. On the other hand, the  $V_i$  are  $U_q(\mathfrak{l}_S)$ -stable since the vectors  $v_i$  are weight vectors. Hence  $V = \sum_i^\oplus V_i$  is a decomposition of  $V$  into irreducible  $U_q(\mathfrak{l}_S)$ -modules.  $\square$

There are obvious notions of weight vectors and weights for  $U_q(\mathfrak{l}_S)$ -modules. With a suitably extended interpretation of the notion of highest weight, the irreducible finite dimensional  $U_q(\mathfrak{l}_S)$ -modules may be characterized in terms of highest weights. We shall only be interested in irreducible  $U_q(\mathfrak{l}_S)$ -modules with weights in the lattice  $P$ . For instance, the restriction of an irreducible  $P$ -weighted  $U_q(\mathfrak{g})$ -module to  $U_q(\mathfrak{l}_S)$  decomposes into such irreducible  $U_q(\mathfrak{l}_S)$ -modules.

Branching rules for the restriction of finite dimensional representations of  $U_q(\mathfrak{g})$  to  $U_q(\mathfrak{l}_S)$  are determined by the behaviour of the corresponding characters. Since the characters for  $P$ -weighted irreducible finite dimensional representations of  $U_q(\mathfrak{g})$  and  $U_q(\mathfrak{l}_S)$  are the same as for the corresponding representations of  $\mathfrak{g}$  and  $\mathfrak{l}_S$ , we have the following proposition.

**PROPOSITION 6.4.2.** *Let  $\lambda \in P_+$ . The multiplicity of any  $P$ -weighted irreducible  $U_q(\mathfrak{l}_S)$ -module in the irreducible decomposition of the restriction of the  $U_q(\mathfrak{g})$ -module  $V(\lambda)$  to  $U_q(\mathfrak{l}_S)$  is the same as in the classical case.*

Next, we define the quantized algebra of functions on  $U/K_S$ . Let  $\iota_S^*: U_q(\mathfrak{g})^* \rightarrow U_q(\mathfrak{l}_S)^*$  be the dual of the Hopf \*-embedding  $\iota_S: U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g})$ , and set

$$\mathbb{C}_q[L_S] := \iota_S^*(\mathbb{C}_q[G]) = \{\phi \circ \iota_S \mid \phi \in \mathbb{C}_q[G]\}.$$

The formulas (6.3.5) uniquely determine a Hopf \*-algebra structure on  $\mathbb{C}_q[L_S]$ , and  $\iota_S^*$  then becomes a Hopf \*-algebra morphism. We write  $\mathbb{C}_q[K_S]$  for  $\mathbb{C}_q[L_S]$  with this



particular choice of  $*$ -structure. Assume now that  $S \neq \Delta$ . Define a  $*$ -subalgebra  $\mathbb{C}_q[U/K_S] \subset \mathbb{C}_q[U]$  by

$$(6.4.4) \quad \begin{aligned} \mathbb{C}_q[U/K_S] &:= \{\phi \in \mathbb{C}_q[U] \mid (\text{id} \otimes \iota_S^*)\Delta(\phi) = \phi \otimes 1\} \\ &= \{\phi \in \mathbb{C}_q[U] \mid X.\phi = \varepsilon(X)\phi, \quad \forall X \in U_q(\mathfrak{t}_S)\} \end{aligned}$$

The algebra  $\mathbb{C}_q[U/K_S]$  is a left  $\mathbb{C}_q[U]$ -subcomodule of  $\mathbb{C}_q[U]$ . We call it the quantized algebra of functions on the generalized flag manifold  $U/K_S$ .

Similarly, the quantized function algebra  $\mathbb{C}_q[K_S^0]$  corresponding to the semisimple part  $K_S^0$  of  $K_S$  can be defined as the image of the dual of the natural embedding  $U_q(\mathfrak{t}_S^0) \hookrightarrow U_q(\mathfrak{g})$ . Its Hopf  $*$ -algebra structure is again given by the formulas (6.3.5). The subalgebra  $\mathbb{C}_q[U/K_S^0]$  then consists by definition of all right  $\mathbb{C}_q[K_S^0]$ -invariant elements in  $\mathbb{C}_q[U]$ . Note that  $\mathbb{C}_q[U/K_S^0] \subset \mathbb{C}_q[U]$  is a left  $U_q(\mathfrak{h})$ -submodule and that  $\mathbb{C}_q[U/K_S]$  coincides with the subalgebra of  $U_q(\mathfrak{h})$ -invariant elements in  $\mathbb{C}_q[U/K_S^0]$ .

We now turn to PBW type factorizations of the algebra  $\mathbb{C}_q[U/K_S]$ . Let  $P(S)$ ,  $P_+(S)$ , respectively  $P_{++}(S)$  be the  $\mathbb{K}$ -span  $\{\varpi_\alpha\}_{\alpha \in S}$  with  $\mathbb{K} = \mathbb{Z}, \mathbb{Z}_+$  respectively  $\mathbb{N}$ . Set  $S^c := \Delta \setminus S$ . The quantized algebra  $A_S^{hol}$  of holomorphic polynomials on  $U/K_S$  is defined by

$$(6.4.5) \quad A_S^{hol} := \bigoplus_{\lambda \in P_+(S^c)} B_\lambda \subset \mathbb{C}_q[U],$$

where  $B_\lambda$  is given by (6.3.12) (cf. [74], [75], [111], [52] and [70]). Note that  $A_S^{hol}$  is a right  $U_q(\mathfrak{g})$ -comodule subalgebra of  $\mathbb{C}_q[U]$ , (6.4.5) being the (multiplicity free) decomposition of  $A_S^{hol}$  into irreducible  $U_q(\mathfrak{g})$ -modules.  $A_S^{hol}$  is generated as algebra by the elements  $C_{\mu, i; \varpi_s}^{\varpi_s}$  ( $\mu \in P(\varpi_s)$ ,  $i \in [1, \dim(V(\varpi_s)_\mu]$ ,  $s \in S^c$ ), which are called the (generalized) quantum holomorphic Plücker coordinates on  $U/K_S$ . The right  $U_q(\mathfrak{g})$ -module algebra  $(A_S^{hol})^* \subset \mathbb{C}_q[U]$  is called the quantized algebra of anti-holomorphic polynomials on  $U/K_S$ . The elements  $(C_{\mu, i; \varpi_s}^{\varpi_s})^*$  ( $\mu \in P(\varpi_s)$ ,  $i \in [1, \dim(V(\varpi_s)_\mu]$ ,  $s \in S^c$ ) are called the (generalized) quantum anti-holomorphic Plücker coordinates.

LEMMA 6.4.3. *The linear subspace*

$$A_S^0 := m((A_S^{hol})^* \otimes A_S^{hol}) \subset \mathbb{C}_q[U],$$

where  $m$  is the multiplication map of  $\mathbb{C}_q[U]$ , is a right  $U_q(\mathfrak{g})$ -submodule  $*$ -subalgebra of  $\mathbb{C}_q[U]$ .

PROOF. Proposition 6.3.2(ii) implies that  $A_S^0$  is a subalgebra of  $\mathbb{C}_q[U]$ . The other assertions are immediate.  $\square$

The subalgebra  $A_S^0$  is generated as algebra by the quantum holomorphic and anti-holomorphic Plücker coordinates on  $U/K_S$ .

REMARK 6.4.4. In the classical setting ( $q = 1$ ), the algebra  $A_S^0$  ( $\#S^c = 1$ ) can be interpreted as algebra of functions on the product of an affine spherical  $G$ -variety with its dual. The  $G$ -module structure on  $A_S^0$  is then related to the doubled  $G$ -action (see [97],

[98] for the terminology). These (and related)  $G$ -varieties have been studied in several papers, see for example [98], [97] and [78].

The algebra  $A_S^0 \subset \mathbb{C}_q[U]$  is stable under the left  $U_q(\mathfrak{h})$ -action, so we can speak of  $U_q(\mathfrak{h})$ -weighted elements in  $A_S^0$ . Let  $A_S$  be the left  $U_q(\mathfrak{h})$ -invariant elements of  $A_S^0$ . Then  $A_S \subset \mathbb{C}_q[U]$  is a right  $U_q(\mathfrak{g})$ -module \*-subalgebra of  $\mathbb{C}_q[U]$ .

LEMMA 6.4.5. *We have  $A_S^0 \subset \mathbb{C}_q[U/K_S^0]$ , so in particular  $A_S \subset \mathbb{C}_q[U/K_S]$ . Furthermore,*

$$(6.4.6) \quad A_S = \text{span}\{(C_{v;v_\lambda}^\lambda)^* C_{w;v_\lambda}^\lambda \mid \lambda \in P_+(S^c), v, w \in V(\lambda)\}.$$

PROOF. Choose  $\lambda \in P_+(S^c)$  and  $i \in S$ . Then we have  $X_i^+ \cdot v_\lambda = 0$  and  $K_i \cdot v_\lambda = v_\lambda$ . It follows that  $\mathbb{C}v_\lambda \subset V(\lambda)$  is a one dimensional  $U_{q_i}(\mathfrak{sl}(2; \mathbb{C}))$ -submodule, where we consider the  $U_{q_i}(\mathfrak{sl}(2; \mathbb{C}))$  action on  $V(\lambda)$  via the embedding  $\phi_i: U_{q_i}(\mathfrak{sl}(2; \mathbb{C})) \hookrightarrow U_q(\mathfrak{g})$ . It follows that  $X_i^- \cdot v_\lambda = 0$ . This readily implies that  $A_S^0 \subset \mathbb{C}_q[U/K_S^0]$ . The remaining assertions are immediate.  $\square$

In view of Lemma 6.4.5, we may consider the subalgebra  $A_S^0$  as the quantum analogue of the algebra of complex-valued polynomial functions on the real manifold  $U/K_S^0$  and the algebra  $A_S$  as the quantum analogue of the algebra of zero-weighted complex-valued polynomials functions on  $U/K_S^0$ . Note that  $A_S$  is generated as algebra by the products  $C_{\mu,i;\varpi_s}^{\varpi_s} (C_{\nu,j;\varpi_s}^{\varpi_s})^* (\mu, \nu \in P(\varpi_s), i \in [1, \dim(V(\varpi_s)_\mu)], j \in [1, \dim(V(\varpi_s)_\nu)]$  and  $s \in S^c$ ).

DEFINITION 6.4.6.  $A_S \subset \mathbb{C}_q[U/K_S]$  is called the factorized \*-subalgebra associated with  $U/K_S$ .

In view of Theorem 6.3.1, there is reason to expect that the factorized algebra  $A_S$  is equal to  $\mathbb{C}_q[U/K_S]$  for any generalized flag manifold  $U/K_S$ . We prove this fact (see Theorem 6.4.10) for a certain subclass of generalized flag manifolds that we shall define and classify in the following proposition. For the proof in these cases we use the so-called Parthasarathy-Ranga Rao-Varadarajan (PRV) conjecture, which was proved independently by Kumar [73] and Mathieu [88]. The PRV conjecture gives information about which irreducible constituents occur in tensor products of irreducible finite dimensional  $\mathfrak{g}$ -modules.

Recall the notations introduced in Section 6.2. The following proposition was observed by Koornwinder [68].

PROPOSITION 6.4.7. ([68]) *Let  $U$  be a connected, simply connected compact Lie group with Lie algebra  $\mathfrak{u}$ , and let  $\mathfrak{p} \subset \mathfrak{g}$  be a standard maximal parabolic subalgebra. Let  $K \subset U$  be the connected subgroup with Lie algebra  $\mathfrak{k} := \mathfrak{p} \cap \mathfrak{u}$ . Then  $(U, K)$  is a Gelfand pair if and only if one of the following three conditions are satisfied:*

- (i)  $(U, K)$  is an irreducible compact Hermitian symmetric pair;
- (ii)  $(U, K) \simeq (SO(2l+1), U(l)), \quad (l \geq 2)$ ;
- (iii)  $(U, K) \simeq (Sp(l), U(1) \times Sp(l-1)), \quad (l \geq 2)$ .

PROOF. For a list of the irreducible compact Hermitian symmetric pairs see [40, Ch. X, Table V]. The proposition follows from this list and the classification of the compact Gelfand pairs  $(U, K)$  with  $U$  simple (cf. [72, Tabelle 1]).  $\square$

Let  $(U, K)$  be a pair from the list (i)–(iii) in Proposition 6.4.7, and let  $(\mathfrak{u}, \mathfrak{k})$  be the associated pair of Lie algebras. Then  $\mathfrak{k} = \mathfrak{k}_S$  for some subset  $S \subset \Delta$  with  $\#S^c = 1$ . We call the simple root  $\alpha \in S^c$  the Gelfand node associated with  $(U, K)$ .

A dominant weight  $\lambda \in P_+$  is called spherical if the the subspace of  $K$ -fixed vectors in  $V(\lambda)$  is one dimensional. The corresponding representation  $V(\lambda)$  is then also called spherical. We write  $P_+^K \subset P_+$  for the subset of dominant spherical weights.

PROPOSITION 6.4.8. *Let  $(U, K)$  be a pair from the list (i)–(iii) in Proposition 6.4.7, and let  $\alpha \in \Delta$  be the associated Gelfand node with corresponding fundamental weight  $\varpi := \varpi_\alpha$ . Then we have a multiplicity free irreducible decomposition of  $U_q(\mathfrak{g})$ -modules of the form*

$$V(\varpi)^* \otimes V(\varpi) \simeq \bigoplus_{i=0}^l V(\mu_i)$$

for certain  $l \in \mathbb{N}$ , where  $\mu_0 := 0 \in P_+$  and  $\{\mu_i\}_{i=1}^l$  is a subset of the dominant spherical weights  $P_+^K$ . Furthermore, every  $\lambda \in P_+^K$  can be uniquely written as a  $\mathbb{Z}_+$ -linear combination of the  $\mu_i$ 's ( $i \in [1, l]$ ).

DEFINITION 6.4.9. *The spherical weights  $\mu_i$  ( $i \in [1, l]$ ) are called the fundamental spherical weights associated with  $(U, K)$ .*

PROOF. It is well known that the trivial representation  $V(0)$  occurs with multiplicity one in the tensor product decomposition of  $V(\varpi)^* \otimes V(\varpi)$ . Furthermore, observe that

$$(6.4.7) \quad V(\varpi)^* \otimes V(\varpi) \simeq (B_\varpi)^* B_\varpi \subset A_{\{\alpha\}^c} \subset \mathbb{C}_q[U/K]$$

as right  $U_q(\mathfrak{g})$ -modules. By Proposition 6.4.2 we have the multiplicity free decomposition as right  $U_q(\mathfrak{g})$ -modules

$$(6.4.8) \quad \mathbb{C}_q[U/K] \simeq \bigoplus_{\lambda \in P_+^K} V(\lambda),$$

from which it follows that the decomposition of  $V(\varpi)^* \otimes V(\varpi)$  is multiplicity free, and that its irreducible constituents are all spherical.

Krämer [72, Tabelle 1] presented for each pair  $(U, K)$  from the list (i)–(iii) in Proposition 6.4.7 a set of dominant spherical weights  $\{\mu_i\}_{i=1}^l$  satisfying the property that every  $\lambda \in P_+^K$  can be uniquely written as a  $\mathbb{Z}_+$ -linear combination of the  $\mu_i$ 's ( $i \in [1, l]$ ). The  $\mu_i$ 's are explicitly given as  $\mathbb{Z}_+$ -linear combination of the fundamental dominant weights  $\varpi_j$  ( $j \in [1, r]$ ). In case of the Hermitean symmetric spaces  $U/K$ , there is an elegant procedure to recover the  $\mu_i$ 's as linear combinations of the fundamental dominant weights from the corresponding Satake diagrams [125].

We show now that all spherical representations  $V(\mu_i)$  ( $i \in [1, l]$ ) are constituents of  $V(\varpi)^* \otimes V(\varpi)$  by using the PRV conjecture, which states the following. Let  $\lambda, \mu \in P_+$  and  $w \in W$ . Let  $[\lambda + w\mu]$  be the unique element in  $P_+$  which lies in the  $W$ -orbit of  $\lambda + w\mu$ . Then  $V([\lambda + w\mu])$  occurs with multiplicity at least one in  $V(\lambda) \otimes V(\mu)$ . For each pair  $(U, K)$  from the list **(i)–(iii)** of Proposition 6.4.7, it is now possible to find explicit Weyl group elements  $w_i \in W$  such that

$$[\varpi - w_i\varpi] = \mu_i, \quad (i = [1, l]).$$

Combined with the PRV conjecture and the fact that  $V(\varpi)^* \simeq V(-\sigma_0\varpi)$ , this implies that  $V(\mu_i)$  is a constituent of  $V(\varpi)^* \otimes V(\varpi)$  for all  $i \in [1, l]$ .

As an example, we follow the procedure for the compact Hermitian symmetric pair  $(U, K) = (SO(2l), U(l))$  ( $l \geq 2$ ). We use the standard realization of the root system  $R$  of type  $D_l$  in the  $l$ -dimensional vector space  $V = \sum_{i=1}^l \mathbb{R}\varepsilon_i$ , with basis given by  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  ( $i = [1, l-1]$ ) and  $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$ . The fundamental weights are given by

$$\begin{aligned} \varpi_i &= \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_i, & (i < l-1), \\ \varpi_{l-1} &= (\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{l-1} - \varepsilon_l)/2, \\ \varpi_l &= (\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{l-1} + \varepsilon_l)/2. \end{aligned}$$

Set  $\varpi = \varpi_l$  (i.e.  $S^c = \{\alpha_l\}$ ). Let  $\sigma_i$  be the linear map defined by  $\varepsilon_j \mapsto -\varepsilon_j$  ( $j = i, i+1$ ) and  $\varepsilon_j \mapsto \varepsilon_j$  otherwise. Then  $\sigma_i \in W$  ( $i = [1, l-1]$ ). If  $l = 2l' + 1$ , then

$$(6.4.9) \quad \begin{aligned} \varpi - \sigma_1\sigma_3 \dots \sigma_{2i-1}\varpi &= \varpi_{2i}, & (i = [1, l'-1]), \\ \varpi - \sigma_1\sigma_3 \dots \sigma_{2l'-1}\varpi &= \varpi_{l-1} + \varpi_l. \end{aligned}$$

If  $l = 2l'$  then we have

$$(6.4.10) \quad \begin{aligned} \varpi - \sigma_1\sigma_3 \dots \sigma_{2i-1}\varpi &= \varpi_{2i}, & (i = [1, l'-1]), \\ \varpi - \sigma_1\sigma_3 \dots \sigma_{2l'-1}\varpi &= 2\varpi_l. \end{aligned}$$

By comparison with [72, Tabelle 1] it follows from (6.4.9) (respectively (6.4.10)) that all the fundamental spherical weights of the pair  $(U, K) = (SO(2l), U(l))$  have been obtained. The other cases are checked in a similar manner.

To complete the proof, we have to show that the  $V(\mu_i)$  ( $i \in [0, l]$ ) are the only irreducible constituents which can occur in the tensor product decomposition of  $V(\varpi)^* \otimes V(\varpi)$ . This is also proved case by case. The cases corresponding to the exceptional groups can be directly verified using for instance the maple-package “qtensor” of Stembridge [115]<sup>1</sup>. The special case  $(U, K) = (SU(p+l), S(U(p) \times U(l)))$  of this proposition was proven in the previous chapter, see (4.5.14). The remaining cases can be checked by showing that for  $\lambda \in P_+^K \setminus \{\mu_i\}_{i=0}^l$ , we have  $\lambda \not\leq \varpi - \sigma_0\varpi$ , which implies that  $V(\lambda)$  cannot occur as constituent of  $V(\varpi)^* \otimes V(\varpi)$ .  $\square$

We are now in the position to prove the main result of this section.

<sup>1</sup><http://www.math.lsa.umich.edu/~jrs/maple.html>

THEOREM 6.4.10. *The factorized \*-subalgebra  $A_S$  is equal to  $\mathbb{C}_q[U/K_S]$  if*

- (i)  $S = \emptyset$ , i.e.  $U/K_S = U/T$  is the full flag manifold;
- (ii)  $\#S^c = 1$  and the simple root  $\alpha \in S^c$  is a Gelfand node.

PROOF. To prove (i) we look at the simultaneous eigenspace decomposition of  $\mathbb{C}_q[U]$  with respect to the left  $U_q(\mathfrak{h})$ -action on  $\mathbb{C}_q[U]$ . The simultaneous eigenspace corresponding to the character  $\varepsilon$  on  $U_q(\mathfrak{h})$  is exactly  $\mathbb{C}_q[U/T]$ . Using Soibelman's factorization of  $\mathbb{C}_q[U]$  (cf. Theorem 6.3.1) and Lemma 6.4.5, it is then easily checked that  $\mathbb{C}_q[U/T] = A_\emptyset$ . To prove (ii) we note that

$$\bigoplus_{i=0}^l V(\mu_i) \simeq (B_\varpi)^* B_\varpi \subset A_S$$

as right  $U_q(\mathfrak{g})$ -modules by Proposition 6.4.8 and (6.4.7) (here we use the notations as introduced in Proposition 6.4.8). Now  $\mathbb{C}_q[U]$  is an integral domain (cf. [49, Lemma 9.1.9 (i)]), hence  $v_\lambda v_\mu \in A_S$  is a highest weight vector of highest weight  $\lambda + \mu$  if  $v_\lambda, v_\mu \in A_S$  are highest weight vectors of highest weight  $\lambda$  respectively  $\mu$ . It follows that

$$\bigoplus_{\lambda \in P_+^K} V(\lambda) \hookrightarrow A_{\{\alpha\}^c}$$

as right  $U_q(\mathfrak{g})$ -modules. Combining with (6.4.8), it follows that  $A_{\{\alpha\}^c} = \mathbb{C}_q[U/K_{\{\alpha\}^c}]$ , as requested.  $\square$

Observe that Theorem 6.4.10 for quantum Grassmannians was already proved in the previous chapter. In fact, for quantum complex Grassmannians we have shown that  $A_S^0 = \mathbb{C}_q[U/K_S^0]$  in Theorem 5.4.1, so in particular  $A_S = \mathbb{C}_q[U/K_S]$ .

REMARK 6.4.11. By yet unpublished results of the author, it turns out that  $A_S$  equals  $\mathbb{C}_q[U/K_S]$  for any generalized flag manifold  $U/K_S$ .

In the remainder of the chapter we study the irreducible \*-representations of the \*-algebras  $A_S$  and  $\mathbb{C}_q[U/K_S]$ . In the next section the restriction of the irreducible \*-representations of  $\mathbb{C}_q[U]$  to the \*-algebras  $A_S$  and  $\mathbb{C}_q[U/K_S]$  is considered first.

### 6.5. Restriction of irreducible \*-representations to $\mathbb{C}_q[U/K]$

We first recall some results from Soibelman [110] concerning the irreducible \*-representations of  $\mathbb{C}_q[U]$ . Let  $\{e_i\}_{i \in \mathbb{Z}_+}$  be the standard orthonormal basis of  $l_2(\mathbb{Z}_+)$ . Write  $\mathcal{B}(l_2(\mathbb{Z}_+))$  for the algebra of bounded linear operators on  $l_2(\mathbb{Z}_+)$ . Then the formulas

$$(6.5.1) \quad \begin{aligned} \pi_q(t_{11})e_j &= \sqrt{(1-q^{2j})}e_{j-1}, & \pi_q(t_{12})e_j &= -q^{j+1}e_j, \\ \pi_q(t_{21})e_j &= q^j e_j, & \pi_q(t_{22})e_j &= \sqrt{(1-q^{2(j+1)})}e_{j+1}, \end{aligned}$$

with  $\pi_q(t_{11})e_0 = 0$ , uniquely determine an irreducible \*-representation

$$\pi_q : \mathbb{C}_q[SU(2)] \rightarrow \mathcal{B}(l_2(\mathbb{Z}_+)).$$

Now the dual of the injective Hopf \*-algebra morphism  $\phi_i : U_{q_i}(\mathfrak{sl}(2; \mathbb{C})) \hookrightarrow U_q(\mathfrak{g})$  corresponding to the  $i$ th node of the Dynkin diagram ( $i \in [1, r]$ ) is a surjective Hopf \*-algebra morphism  $\phi_i^* : \mathbb{C}_q[U] \rightarrow \mathbb{C}_{q_i}[SU(2)]$ . In particular, we obtain irreducible \*-representations  $\pi_i := \pi_{q_i} \circ \phi_i^* : \mathbb{C}_q[U] \rightarrow \mathcal{B}(l_2(\mathbb{Z}_+))$ .

On the other hand, there is a family of one dimensional \*-representations  $\tau_t$  of  $\mathbb{C}_q[U]$  parametrized by the maximal torus  $t \in T \simeq \mathbb{T}^r$  ( $\mathbb{T} \subset \mathbb{C}$  denoting the unit circle in the complex plane). More explicitly, let  $\iota_T : U_q(\mathfrak{h}) \hookrightarrow U_q(\mathfrak{g})$  be the natural Hopf \*-algebra embedding, and set  $\mathbb{C}_q[T] := \text{span}\{\phi_\mu\}_{\mu \in P} \subset U_q(\mathfrak{h})^*$ , where  $\phi_\mu(K^\sigma) := q^{(\mu, \sigma)}$  for  $\sigma \in Q$ . The formulas (6.3.5) define a Hopf \*-algebra structure on  $\mathbb{C}_q[T]$ . Then  $\iota_T^* : \mathbb{C}_q[U] \rightarrow \mathbb{C}_q[T]$ ,  $\iota_T^*(\phi) := \phi \circ \iota_T$  is a surjective Hopf \*-algebra morphism. Any irreducible \*-representation of  $\mathbb{C}_q[T]$  is one dimensional and can be written as  $\tilde{\tau}_t(\phi_\mu) := t^\mu$  for a unique  $t \in T \simeq \mathbb{T}^r$ . Here  $t^\mu := t_1^{m_1} \dots t_r^{m_r}$  for  $\mu = \sum_{i=1}^r m_i \varpi_i$ . So we obtain a one dimensional \*-representation  $\tau_t := \tilde{\tau}_t \circ \iota_T^*$  of  $\mathbb{C}_q[U]$ , which is given explicitly on matrix elements  $C_{\mu, i; \nu, j}^\lambda$  by the formula

$$(6.5.2) \quad \tau_t(C_{\mu, i; \nu, j}^\lambda) = \delta_{\mu, \nu} \delta_{i, j} t^\mu.$$

The following theorem completely describes the irreducible \*-representations of  $\mathbb{C}_q[U]$ .

**THEOREM 6.5.1 (Soibelman [110]).** *Let  $\sigma \in W$ , and fix a reduced expression  $\sigma = s_{i_1} s_{i_2} \dots s_{i_l}$ . The \*-representation*

$$(6.5.3) \quad \pi_\sigma := \pi_{i_1} \otimes \pi_{i_2} \otimes \dots \otimes \pi_{i_l}$$

*does not depend on the choice of reduced expression (up to equivalence). The set*

$$\{\pi_\sigma \otimes \tau_t \mid t \in T, \sigma \in W\}$$

*is a complete set of mutually inequivalent irreducible \*-representations of  $\mathbb{C}_q[U]$ .*

Here tensor products of \*-representations are defined in the usual way by means of the coalgebra structure on  $\mathbb{C}_q[U]$ . The irreducible representation  $\pi_e$  with respect to the unit element  $e \in W$  is the one dimensional \*-representation associated with the counit  $\varepsilon$  on  $\mathbb{C}_q[U]$ . In Soibelman's terminology, the representations  $\pi_\sigma \otimes \tau_t$  are said to be associated with the Schubert cell  $X_\sigma$  of  $U/T$  (cf. Section 6.2).

We also mention here an important property of the kernel of  $\pi_\sigma$ , which we will repeatedly need later on. Let  $U_q(\mathfrak{b}_+)$  be the subalgebra of  $U_q(\mathfrak{g})$  generated by the  $K_i^{\pm 1}$  and the  $X_i^+$  ( $i \in [1, r]$ ). For any  $\lambda \in P_+$ , the \*-representation  $\pi_\sigma$  satisfies

$$(6.5.4) \quad \pi_\sigma(C_{v; v_\lambda}^\lambda) = 0 \quad (v \notin U_q(\mathfrak{b}_+)v_\sigma), \quad \pi_\sigma(C_{v_\sigma; v_\lambda}^\lambda) \neq 0$$

(cf. [110, Theorem 5.7]). Formula (6.5.4) combined with [9, Lemma 2.12] shows that the classical limit of the kernel of  $\pi_\sigma$  formally tends to the ideal of functions vanishing on  $X_\sigma$ .

Fix now a subset  $S \subsetneq \Delta$ . We freely use the notations introduced earlier in this chapter. Our next goal is to describe how the \*-representations  $\pi_\sigma$  decompose under restriction to the subalgebra  $\mathbb{C}_q[U/K_S]$ . Consider the selfadjoint operators

$$(6.5.5) \quad L_{\sigma\lambda; \lambda} := \pi_\sigma((C_{\sigma\lambda; \lambda}^\lambda)^* C_{\sigma\lambda; \lambda}^\lambda)$$

for  $\lambda \in P_+(S^c)$ . Let  $\sigma = s_{i_1} \cdots s_{i_l}$  be a reduced expression for  $\sigma$ , and set  $\pi_\sigma = \pi_{i_1} \otimes \pi_{i_2} \otimes \cdots \otimes \pi_{i_l}$ . Then it follows from [110, Proof of Proposition 5.2] (see also [110, Proof of Proposition 5.8]) that

$$(6.5.6) \quad \pi_\sigma(C_{\sigma\lambda}^\lambda) = c \pi_{q_{i_1}}(t_{21})^{(\lambda, \gamma_1^\vee)} \otimes \pi_{q_{i_2}}(t_{21})^{(\lambda, \gamma_2^\vee)} \otimes \cdots \otimes \pi_{q_{i_l}}(t_{21})^{(\lambda, \gamma_l^\vee)}$$

where the scalar  $c \in \mathbb{T}$  depends on the particular choices of bases for the irreducible representations  $V(\mu)$  ( $\mu \in P_+$ ), and with

$$(6.5.7) \quad \gamma_k := s_{i_l} s_{i_{l-1}} \cdots s_{i_{k+1}}(\alpha_{i_k}) \quad (1 \leq k \leq l-1), \quad \gamma_l := \alpha_{i_l}.$$

The proof of (6.5.6), which was given in [110] under the assumption that  $\lambda \in P_{++}$ , is in fact valid for all dominant weights  $\lambda \in P_+$ . It follows from (6.5.1), (6.5.5) and (6.5.6) that  $l_2(\mathbb{Z}_+)^{\otimes l(\sigma)}$  decomposes as an orthogonal direct sum of eigenspaces for  $L_{\sigma\lambda; \lambda}$ ,

$$(6.5.8) \quad l_2(\mathbb{Z}_+)^{\otimes l(\sigma)} = \bigoplus_{\gamma \in I(\lambda)} H_\gamma(\lambda),$$

where  $I(\lambda) \subset (0, 1]$  denotes the set of eigenvalues of  $L_{\sigma\lambda; \lambda}$ , and  $H_\gamma(\lambda)$  denotes the eigenspace of  $L_{\sigma\lambda; \lambda}$  corresponding to the eigenvalue  $\gamma \in I(\lambda)$  (we suppress the dependence on  $\sigma$  if there is no confusion possible). Observe that  $1 \in I(\lambda)$  and that  $L_{\sigma\lambda; \lambda}$  is injective.

Recall the definition of the set  $W^S$  of minimal coset representatives (cf. (6.2.10)). An alternative characterization of  $W^S$  is given by

$$(6.5.9) \quad W^S = \{\sigma \in W \mid \sigma(R_S^+) \subset R^+\},$$

where  $R_S^+ := R^+ \cap \text{span}\{S\}$  (cf. [9, Proposition 5.1 (iii)]). Using this alternative description of  $W^S$  we obtain the following properties of  $L_{\sigma\lambda; \lambda}$  for  $\lambda \in P_{++}(S^c)$ .

**PROPOSITION 6.5.2.** *Suppose that  $\sigma \in W^S$  and  $\lambda \in P_{++}(S^c)$ . Then*

- (i)  $L_{\sigma\lambda; \lambda}$  is a compact operator;
- (ii) The eigenspace  $H_1(\lambda)$  of  $L_{\sigma\lambda; \lambda}$  corresponding to the eigenvalue 1 is spanned by the vector  $e_0^{\otimes l(\sigma)}$ .

**PROOF.** Fix a  $\lambda \in P_{++}(S^c)$ , and let  $\sigma = s_{i_1} s_{i_2} \cdots s_{i_l}$  be a reduced expression of a minimal coset representative  $\sigma \in W^S$ . It is well-known that

$$(6.5.10) \quad R^+ \cap \sigma^{-1}(R^-) = \{\gamma_k\}_{k=1}^l,$$

where the  $\gamma_k$  are defined by (6.5.7). We have  $\gamma_k \in R^+ \setminus R_S^+$  by (6.5.9). It follows that  $(\lambda, \gamma_k^\vee) > 0$  for all  $k$ , since  $\lambda \in P_{++}(S^c)$ . By (6.5.1) and (6.5.6) it follows that  $H_1(\lambda) = \text{span}\{e_0^{\otimes l(\sigma)}\}$  and that  $H_\gamma(\lambda)$  is finite dimensional for all  $\gamma \in I(\lambda)$ . Since the spectrum of  $L_{\sigma\lambda; \lambda}$  (which is equal to  $I(\lambda) \cup \{0\}$ ) does not have a limit point except 0, we conclude that  $L_{\sigma\lambda; \lambda}$  is a compact operator (cf. [105, Theorem 12.30]).  $\square$

We recall now the following well-known inequalities for weights of finite dimensional irreducible representations of  $\mathfrak{g}$  (or, equivalently,  $U_q(\mathfrak{g})$ ).

PROPOSITION 6.5.3. *Let  $\lambda \in P_+$  and  $\mu, \nu \in P(\lambda)$ . Then  $(\lambda, \lambda) \geq (\mu, \nu)$ , and equality holds if and only if  $\mu = \nu \in W\lambda$ .*

For a proof of the proposition, see for instance [53, Proposition 11.4]. The proof is based on the following lemma, which we will also need later on. The lemma is a slightly weaker version of [53, Lemma 11.2].

LEMMA 6.5.4. *Let  $\lambda \in P_+$  and  $\mu \in P(\lambda) \setminus \{\lambda\}$ , and let  $m_i \in \mathbb{Z}_+$  ( $i \in [1, r]$ ) be the expansion coefficients defined by  $\lambda - \mu = \sum_i m_i \alpha_i$ . Then there is an  $1 \leq i \leq r$  with  $m_i > 0$  and  $\lambda(H_i) \neq 0$ .*

We now have the following proposition, which can be regarded as a quantum analogue of the “if” part of Proposition 6.2.2.

PROPOSITION 6.5.5. *Let  $\sigma \in W^S$ . Then  $\pi_\sigma$  restricts to an irreducible \*-representation of the factorized \*-algebra  $A_S$ . In particular,  $\pi_\sigma$  restricts to an irreducible \*-representation of  $\mathbb{C}_q[U/K_S]$ .*

PROOF. Let  $\lambda \in P_{++}(S^c)$  and  $\sigma \in W^S$ . Suppose  $H \subset l_2(\mathbb{Z}_+)^{\otimes l(\sigma)}$  is a non-zero closed subspace invariant under  $\pi_{\sigma|_{A_S}}$ . Set  $\gamma := \|L_{\sigma\lambda; \lambda}|_H\|$ . Then  $\gamma > 0$  since  $L_{\sigma\lambda; \lambda}$  is injective, and  $\gamma$  is an eigenvalue of  $L_{\sigma\lambda; \lambda}|_H$  by Proposition 6.5.2(i). Let  $H_\gamma$  be the corresponding eigenspace. We claim that

$$(6.5.11) \quad \pi_\sigma((C_{\mu, i; \lambda}^\lambda)^* C_{\mu, i; \lambda}^\lambda) H_\gamma = 0, \quad \mu \neq \sigma\lambda.$$

Suppose for the moment that the claim is correct. Then (6.3.10) and (6.5.11) imply  $\gamma = 1$ , hence  $H_\gamma = \text{span}\{e_0^{\otimes l(\sigma)}\}$  by Proposition 6.5.2(ii). So every non-zero closed invariant subspace contains the vector  $e_0^{\otimes l(\sigma)}$ . Since  $H^\perp$  is also a closed invariant subspace, we must have  $H^\perp = \{0\}$ , i.e.  $H = l_2(\mathbb{Z}_+)^{\otimes l(\sigma)}$ . Remains therefore to prove the claim (6.5.11). By (6.5.4) we have  $\pi_\sigma(C_{\mu, i; \lambda}^\lambda) = 0$  if  $\mu < \sigma\lambda$ . Hence

$$\begin{aligned} L_{\sigma\lambda; \lambda} \pi_\sigma((C_{\mu, i; \lambda}^\lambda)^* C_{\mu, i; \lambda}^\lambda) &= q^{(\lambda, \lambda) - (\mu, \sigma\lambda)} \pi_\sigma((C_{\mu, i; \lambda}^\lambda C_{\sigma\lambda; \lambda}^\lambda)^* C_{\sigma\lambda; \lambda}^\lambda C_{\mu, i; \lambda}^\lambda) \\ &= q^{2(\lambda, \lambda) - 2(\mu, \sigma\lambda)} \pi_\sigma((C_{\mu, i; \lambda}^\lambda)^* (C_{\sigma\lambda; \lambda}^\lambda)^* C_{\sigma\lambda; \lambda}^\lambda C_{\mu, i; \lambda}^\lambda) \\ &= q^{2(\lambda, \lambda) - 2(\mu, \sigma\lambda)} \pi_\sigma((C_{\mu, i; \lambda}^\lambda)^* C_{\sigma\lambda; \lambda}^\lambda) \pi_\sigma((C_{\sigma\lambda; \lambda}^\lambda)^* C_{\mu, i; \lambda}^\lambda), \end{aligned}$$

where we used Proposition 6.3.2(i) in the second equality and Proposition 6.3.2(ii) in the first and third equality. So (6.5.11) will then follow from

$$(6.5.12) \quad \pi_\sigma((C_{\sigma\lambda; \lambda}^\lambda)^* C_{\mu, i; \lambda}^\lambda) H_\gamma = 0, \quad \mu \neq \sigma\lambda,$$

in view of the injectivity of  $L_{\sigma\lambda; \lambda}$ . Fix  $h \in H_\gamma$  and  $\mu \in P(\lambda)$  with  $\mu \neq \sigma\lambda$ . By Lemma 6.4.5 we have  $(C_{\sigma\lambda; \lambda}^\lambda)^* C_{\mu, i; \lambda}^\lambda \in A_S \subset \mathbb{C}_q[U/K_S]$ , hence the vector

$$(6.5.13) \quad \tilde{h} := \pi_\sigma((C_{\sigma\lambda; \lambda}^\lambda)^* C_{\mu, i; \lambda}^\lambda) h$$

lies in the invariant subspace  $H$ . Again using the commutation relations given in Proposition 6.3.2 and Corollary 6.3.3, we see that  $\tilde{h}$  is an eigenvector of  $L_{\sigma\lambda; \lambda}$  with eigenvalue  $\tilde{\gamma} := q^{2(\lambda, \sigma^{-1}(\mu) - \lambda)} \gamma$ . We have  $\tilde{\gamma} > \gamma$  by Proposition 6.5.3. By the maximality of  $\gamma$ , we conclude that  $\tilde{h} = 0$ . This proves (6.5.12), hence also the claim (6.5.11).  $\square$



DEFINITION 6.5.6. *The irreducible \*-representation  $\pi_\sigma$  ( $\sigma \in W^S$ ) of  $\mathbb{C}_q[U/K_S]$  is said to be associated with the Schubert cell  $X_{\bar{\sigma}} \subset U/K_S$ .*

The following proposition can be regarded as a quantum analogue of Proposition 6.2.3 as well as of the “only if” part of Proposition 6.2.2.

PROPOSITION 6.5.7. *Let  $\sigma \in W$ , and let  $\sigma = uv$  be the unique decomposition of  $\sigma$  with  $u \in W^S$  and  $v \in W_S$ . For  $\pi_\sigma = \pi_u \otimes \pi_v$  (cf. (6.2.11)) and  $t \in T$ , we have*

$$(\pi_\sigma \otimes \tau_t)(a) = \pi_u(a) \otimes \text{id}^{\otimes l(v)}, \quad a \in \mathbb{C}_q[U/K_S].$$

PROOF. Recall that the one dimensional \*-representation  $\tau_t$  factorizes through  $\iota_T^* : \mathbb{C}_q[U] \rightarrow \mathbb{C}_q[T]$  and that  $\pi_i$  factorizes through  $\phi_i^* : \mathbb{C}_q[U] \rightarrow \mathbb{C}_{q_i}[SU(2)]$ . The maps  $\iota_T^*$  and  $\phi_i^*$  ( $i \in S$ ) factorize through  $\iota_S^* : \mathbb{C}_q[U] \rightarrow \mathbb{C}_q[K_S]$  since the ranges of  $\iota_T$  and  $\phi_i$  ( $i \in S$ ) lie in the Hopf-subalgebra  $U_q(\mathfrak{g}_S)$ . Hence  $\pi_v \otimes \tau_t$  ( $v \in W_S, t \in T$ ) factorizes through  $\iota_S^*$ , say  $\pi_v \otimes \tau_t = \pi_{v,t} \circ \iota_S^*$ . Then we have for  $a \in \mathbb{C}_q[U/K_S]$ ,

$$\begin{aligned} (\pi_\sigma \otimes \tau_t)(a) &= (\pi_u \otimes \pi_v \otimes \tau_t) \circ \Delta(a) \\ &= (\pi_u \otimes \pi_{v,t}) \circ (\text{id} \otimes \iota_S^*) \Delta(a) \\ &= \pi_u(a) \otimes \pi_{v,t}(1) = \pi_u(a) \otimes \text{id}^{\otimes l(v)}, \end{aligned}$$

which completes the proof of the proposition.  $\square$

LEMMA 6.5.8. *The \*-representations  $\{\pi_\sigma\}_{\sigma \in W^S}$ , considered as \*-representations of  $A_S$  respectively  $\mathbb{C}_q[U/K_S]$ , are mutually inequivalent.*

PROOF. Let  $\sigma, \sigma' \in W^S$  with  $\sigma \neq \sigma'$  and  $\lambda \in P_{++}(S^c)$ . Then  $\sigma\lambda \neq \sigma'\lambda$ , since the isotropy subgroup  $\{\sigma \in W \mid \sigma\lambda = \lambda\}$  is equal to  $W_S$  by Chevalley’s Lemma (cf. [58, Proposition 2.72]). Without loss of generality we may assume that  $\sigma\lambda \not\geq \sigma'\lambda$ . Then we have  $\pi_{\sigma'}((C_{\sigma\lambda,\lambda}^\lambda)^* C_{\sigma\lambda,\lambda}^\lambda) = 0$  by (6.5.4). On the other hand,  $L_{\sigma\lambda,\lambda}$  is injective. It follows that  $\pi_\sigma \not\sim \pi_{\sigma'}$  as \*-representations of  $A_S$ .  $\square$

Let now  $\|\cdot\|_u$  be the universal  $C^*$ -norm on  $\mathbb{C}_q[U]$  (cf. [23, Section 4]), so

$$(6.5.14) \quad \|a\|_u := \sup_{\sigma \in W, t \in T} \|(\pi_\sigma \otimes \tau_t)(a)\|, \quad a \in \mathbb{C}_q[U].$$

Let  $C_q(U)$  (respectively  $C_q(U/K_S)$ ) be the completion of the \*-algebra  $\mathbb{C}_q[U]$  (respectively of the \*-algebra  $\mathbb{C}_q[U/K_S]$ ) with respect to  $\|\cdot\|_u$ . All \*-representations  $\pi_\sigma \otimes \tau_t$  of  $\mathbb{C}_q[U]$  extend to \*-representations of the  $C^*$ -algebra  $C_q(U)$  by continuity. The results of this section can now be summarized as follows.

THEOREM 6.5.9. *Let  $S \subsetneq \Delta$ . Then  $\{\pi_\sigma\}_{\sigma \in W^S}$  is a complete set of mutually inequivalent irreducible \*-representations of  $C_q(U/K_S)$ .*

PROOF. This follows from the previous results, since every irreducible \*-representation of  $C_q(U/K_S)$  appears as an irreducible component of  $\sigma|_{C_q(U/K_S)}$  for some irreducible \*-representation  $\sigma$  of  $C_q(U)$  (cf. [26, Proposition 2.10.2]).  $\square$

Theorem 6.5.9 does not imply that  $\{\pi_\sigma\}_{\sigma \in W^S}$  is a complete set of irreducible \*-representations of the \*-algebra  $\mathbb{C}_q[U/K_S]$  itself. Indeed, it is not clear that any irreducible \*-representation of  $\mathbb{C}_q[U/K_S]$  can be continuously extended to a \*-representation of  $C_q(U/K_S)$ . In the remainder of this chapter we will deal with the classification of the irreducible \*-representations of  $A_S$ . In particular, this will yield a complete classification of the irreducible \*-representations of  $\mathbb{C}_q[U/K_S]$  for the generalized flag manifolds  $U/K_S$  for which the PBW factorization is valid (cf. Theorem 6.4.10 and Remark 6.4.11).

### 6.6. Irreducible \*-representations of $A_S$

Let  $S \subsetneq \Delta$  be any subset. In this section it is shown that  $\{\pi_\sigma\}_{\sigma \in W^S}$  exhausts the set of irreducible \*-representations of  $A_S$  (up to equivalence). We fix therefore an arbitrary irreducible \*-representation  $\tau : A_S \rightarrow \mathcal{B}(H)$  and we will show that  $\tau \simeq \pi_\sigma$  for a (unique)  $\sigma \in W^S$ . In order to associate the proper minimal coset representative  $\sigma \in W^S$  with  $\tau$ , we need to study the range  $\tau(A_S) \subset \mathcal{B}(H)$  of  $\tau$  in more detail. For  $\lambda \in P_+(S^c)$  and  $\mu, \nu \in P(\lambda)$ , let  $\tau^\lambda(\mu; \nu), \tau^\lambda(\nu) \subset \mathcal{B}(H)$  be the linear subspaces

$$(6.6.1) \quad \begin{aligned} \tau^\lambda(\mu; \nu) &:= \{\tau((C_{v;v_\lambda}^\lambda)^* C_{w;v_\lambda}^\lambda) \mid v \in V(\lambda)_\mu, w \in V(\lambda)_\nu\}, \\ \tau^\lambda(\nu) &:= \{\tau((C_{v;v_\lambda}^\lambda)^* C_{w;v_\lambda}^\lambda) \mid v \in V(\lambda), w \in V(\lambda)_\nu\}. \end{aligned}$$

For  $\lambda \in P_+(S^c)$ , set

$$(6.6.2) \quad D(\lambda) := \{\nu \in P(\lambda) \mid \tau^\lambda(\nu) \neq \{0\}\}$$

and let  $D_m(\lambda)$  be the set of weights  $\nu \in D(\lambda)$  such that  $\nu' \notin D(\lambda)$  for all  $\nu' < \nu$ . By (6.3.10), we have  $D(\lambda) \neq \emptyset$ , hence also  $D_m(\lambda) \neq \emptyset$ . We start with a lemma which is useful for the computation of commutation relations in  $\tau(A_S) \subset \mathcal{B}(H)$ .

**LEMMA 6.6.1.** *Let  $\lambda, \Lambda \in P_+(S^c)$  and  $\nu \in D_m(\lambda)$ . Let  $v \in V(\lambda)$ ,  $v' \in V(\lambda)_{\nu'}$  with  $\nu' < \nu$  and  $w, w' \in V(\Lambda)$ . Then the product of the four matrix elements  $(C_{v;v_\lambda}^\lambda)^*$ ,  $C_{v';v_\lambda}^\lambda$ ,  $(C_{w;v_\Lambda}^\Lambda)^*$ , and  $C_{w';v_\Lambda}^\Lambda$ , taken in an arbitrary order, is contained in  $\text{Ker}(\tau)$ .*

**PROOF.** Since  $\text{Ker}(\tau)$  is a two-sided \*-ideal in  $A_S$ , it follows from the definitions that

$$(C_{w;v_\Lambda}^\Lambda)^* C_{w';v_\Lambda}^\Lambda (C_{v;v_\lambda}^\lambda)^* C_{v';v_\lambda}^\lambda \in \text{Ker}(\tau).$$

If the product of the four matrix coefficients is taken in a different order, then we can rewrite it by Proposition 6.3.2 and by Corollary 6.3.3 as a linear combination of products of matrix elements

$$(C_{u;v_\Lambda}^\Lambda)^* C_{u';v_\Lambda}^\Lambda (C_{x;v_\lambda}^\lambda)^* C_{x';v_\lambda}^\lambda$$

with  $x' \in V(\lambda)_{\nu''}$  and  $\nu'' \leq \nu' < \nu$ . These are all contained in  $\text{Ker}(\tau)$ , since  $\nu \in D_m(\lambda)$ .  $\square$

**LEMMA 6.6.2.** *Let  $\lambda \in P_+(S^c)$  and  $\nu \in D_m(\lambda)$ . Then*

- (i)  $\tau^\lambda(\nu; \nu) \neq \{0\}$ ;
- (ii)  $\nu = \sigma\lambda$  for some  $\sigma \in W^S$ .

PROOF. Let  $\lambda \in P_+(S^c)$  and  $\nu \in D_m(\lambda)$ . Fix weight vectors  $v \in V(\lambda)_\mu$ ,  $w \in V(\lambda)_\nu$  such that  $T_{v;w} := \tau((C_{v;v_\lambda}^\lambda)^* C_{w;v_\lambda}^\lambda) \neq 0$ . By Lemma 6.6.1, we compute

$$(6.6.3) \quad \begin{aligned} (T_{v;w})^* T_{v;w} &= q^{(\mu,\nu)-(\lambda,\lambda)} \tau(C_{v;v_\lambda}^\lambda (C_{v;v_\lambda}^\lambda C_{w;v_\lambda}^\lambda)^* C_{w;v_\lambda}^\lambda) \\ &= \tau(C_{v;v_\lambda}^\lambda (C_{v;v_\lambda}^\lambda)^*) T_{w;w}, \end{aligned}$$

where Proposition 6.3.2(ii) is used in the first equality and Proposition 6.3.2(i) is used in the second equality. On the other hand,  $(T_{v;w})^* T_{v;w} \neq 0$  since  $\mathcal{B}(H)$  is a  $C^*$ -algebra, so it follows that  $T_{w;w} \neq 0$ . In particular,  $\tau^\lambda(\nu, \nu) \neq \{0\}$ . Formula (6.6.3) for  $v = w$  gives

$$0 \neq (T_{w;w})^* T_{w;w} = \tau(C_{w;v_\lambda}^\lambda (C_{w;v_\lambda}^\lambda)^*) T_{w;w} = q^{(\lambda,\lambda)-(\nu,\nu)} T_{w;w} T_{w;w},$$

where Proposition 6.3.2(ii) is used in the last equality. It follows that  $(\lambda, \lambda) = (\nu, \nu)$ , since  $T_{w;w}$  is selfadjoint. By Proposition 6.5.3 we obtain  $\nu = \sigma\lambda$  for some  $\sigma \in W^S$ .  $\square$

For  $\lambda \in P_+(S^c)$  and  $\nu \in D_m(\lambda)$ , we set

$$(6.6.4) \quad L_{\nu;\lambda} := \tau((C_{\nu;\lambda}^\lambda)^* C_{\nu;\lambda}^\lambda).$$

This definition makes sense since  $\dim(V(\lambda)_\nu) = 1$  by Lemma 6.6.2(ii). Furthermore,  $L_{\nu;\lambda}$  is a non-zero selfadjoint operator which commutes with the elements of  $\tau(A_S)$  in the following way.

LEMMA 6.6.3. *Let  $\lambda, \Lambda \in P_+(S^c)$  and  $\nu \in D_m(\lambda)$ . For  $v \in V(\Lambda)_\mu$ ,  $w \in V(\Lambda)_{\mu'}$  we have*

$$L_{\nu;\lambda} \tau((C_{v;v_\Lambda}^\Lambda)^* C_{w;v_\Lambda}^\Lambda) = q^{2(\nu,\mu'-\mu)} \tau((C_{v;v_\Lambda}^\Lambda)^* C_{w;v_\Lambda}^\Lambda) L_{\nu;\lambda}.$$

PROOF. By Lemma 6.6.1 and the commutation relations in Section 6.3 we compute

$$\begin{aligned} L_{\nu;\lambda} \tau((C_{v;v_\Lambda}^\Lambda)^* C_{w;v_\Lambda}^\Lambda) &= q^{(\lambda,\Lambda)-(\nu,\mu)} \tau((C_{v;v_\Lambda}^\Lambda C_{v_\nu;v_\lambda}^\lambda)^* C_{v_\nu;v_\lambda}^\lambda C_{w;v_\Lambda}^\Lambda) \\ &= q^{2(\lambda,\Lambda)-2(\nu,\mu)} \tau((C_{v;v_\Lambda}^\Lambda)^* (C_{v_\nu;v_\lambda}^\lambda)^* C_{v_\nu;v_\lambda}^\lambda C_{w;v_\Lambda}^\Lambda) \\ &= q^{(\nu,\mu')+(\lambda,\Lambda)-2(\nu,\mu)} \tau((C_{v;v_\Lambda}^\Lambda)^* (C_{v_\nu;v_\lambda}^\lambda)^* C_{w;v_\Lambda}^\Lambda C_{v_\nu;v_\lambda}^\lambda) \\ &= q^{2(\nu,\mu'-\mu)} \tau((C_{v;v_\Lambda}^\Lambda)^* C_{w;v_\Lambda}^\Lambda) L_{\nu;\lambda}, \end{aligned}$$

where Proposition 6.3.2(ii) is used for the first and fourth equality, Proposition 6.3.2(i) is used for the second equality, and Corollary 6.3.3 is used for the third equality.  $\square$

It follows from Lemma 6.6.3 that  $\text{Ker}(L_{\nu;\lambda}) \subsetneq H$  is a closed invariant subspace. By the irreducibility of  $\tau$ , we thus obtain the following corollary.

COROLLARY 6.6.4. *Let  $\lambda \in P_+(S^c)$  and  $\nu \in D_m(\lambda)$ . Then  $L_{\nu;\lambda}$  is injective.*

The minimal coset representative  $\sigma$  of Lemma 6.6.2(ii) is unique and independent of  $\lambda \in P_+(S^c)$  in the following sense.

LEMMA 6.6.5. *There exists a unique  $\sigma \in W^S$  such that  $D_m(\lambda) = \{\sigma\lambda\}$  for all  $\lambda \in P_+(S^c)$ .*

PROOF. Let  $\Lambda \in P_{++}(S^c)$  and  $\nu \in D_m(\Lambda)$ . Then there exists a unique  $\sigma \in W^S$  such that  $\nu = \sigma\Lambda$  by Lemma 6.6.2(ii) and by Chevalley's Lemma (cf. [58, Proposition 2.27]). Fix furthermore arbitrary  $\lambda \in P_+(S^c)$  and  $\nu' \in D_m(\lambda)$ . Choose a  $\sigma' \in W$  such that  $\nu' = \sigma'\lambda$ . By Lemma 6.6.1 and the commutation relations of Section 6.3, we compute

$$\begin{aligned} L_{\nu;\Lambda} L_{\nu';\lambda} &= q^{(\Lambda,\lambda) - (\nu,\nu')} \tau((C_{\nu';\lambda}^\lambda C_{\nu;\Lambda}^\Lambda)^* C_{\nu;\Lambda}^\Lambda C_{\nu';\lambda}^\lambda) \\ &= q^{3(\Lambda,\lambda) - 3(\nu,\nu')} \tau((C_{\nu';\lambda}^\lambda)^* (C_{\nu;\Lambda}^\Lambda)^* C_{\nu';\lambda}^\lambda C_{\nu;\Lambda}^\Lambda) \\ &= q^{2(\Lambda,\lambda) - 2(\nu,\nu')} L_{\nu';\lambda} L_{\nu;\Lambda}, \end{aligned}$$

where Proposition 6.3.2(ii) is used in the first and third equality and Proposition 6.3.2(i) is used twice in the second equality. If we repeat the same computation, but now using Corollary 6.3.3 twice in the second equality, then we obtain

$$L_{\nu;\Lambda} L_{\nu';\lambda} = q^{2(\nu,\nu') - 2(\Lambda,\lambda)} L_{\nu';\lambda} L_{\nu;\Lambda},$$

hence

$$(q^{2(\Lambda,\lambda) - 2(\nu,\nu')} - q^{2(\nu,\nu') - 2(\Lambda,\lambda)}) L_{\nu';\lambda} L_{\nu;\Lambda} = 0.$$

By Corollary 6.6.4 we have  $L_{\nu';\lambda} L_{\nu;\Lambda} \neq 0$ , so we conclude that

$$(\Lambda, \lambda) - (\nu, \nu') = (\Lambda, \lambda - \sigma^{-1}\sigma'\lambda) = 0.$$

Since  $\Lambda \in P_{++}(S^c)$  and  $\lambda \in P_+(S^c)$ , it follows from Lemma 6.5.4 that  $\lambda = \sigma^{-1}\sigma'\lambda$ , i.e.  $\nu' = \sigma\lambda$ . Hence,  $D_m(\lambda) = \{\sigma\lambda\}$  for all  $\lambda \in P_+(S^c)$ .  $\square$

In the remainder of this section we write  $\sigma$  for the unique minimal coset representative such that  $D_m(\lambda) = \{\sigma\lambda\}$  for all  $\lambda \in P_+(S^c)$ . We are going to prove that  $\tau \simeq \pi_\sigma$ . First we look for the analogue of the distinguished vector  $e_0^{\otimes I(\sigma)}$  (cf. Proposition 6.5.2(ii)) in the representation space  $H$  of  $\tau$ .

The spectrum  $I(\lambda)$  of  $L_{\sigma\lambda;\lambda}$  is contained in  $[0, \infty)$ , since  $L_{\sigma\lambda;\lambda}$  is a positive operator. By considering the spectral decomposition of  $L_{\sigma\lambda;\lambda}$ , the following corollary of Lemma 6.6.3 and [60, Lemma 4.3] is obtained.

**COROLLARY 6.6.6.** *Let  $\lambda \in P_+(S^c)$ . Then  $I(\lambda) \subset [0, \infty)$  is a countable set with no limit points, except possibly 0.*

The proof of Corollary 6.6.6 is similar to the proof of [110, Proposition 3.9] and of [60, Proposition 4.2].

By Corollary 6.6.6 we have an orthogonal direct sum decomposition

$$(6.6.5) \quad H = \bigoplus_{\gamma \in I(\lambda) \cap \mathbb{R}_{>0}} H_\gamma(\lambda)$$

into eigenspaces of  $L_{\sigma\lambda;\lambda}$ , where  $H_\gamma(\lambda)$  is the eigenspace of  $L_{\sigma\lambda;\lambda}$  corresponding to the eigenvalue  $\gamma$ . Let  $\gamma_0(\lambda) > 0$  be the largest eigenvalue of  $L_{\sigma\lambda;\lambda}$ .

**LEMMA 6.6.7.** *Let  $\lambda \in P_+(S^c)$ ,  $v \in V(\lambda)$ ,  $w \in V(\lambda)_\nu$  and assume that  $\nu \neq \sigma\lambda$ . Then  $\tau((C_{v;\nu\lambda}^\lambda)^* C_{w;\nu\lambda}^\lambda)(H_{\gamma_0(\lambda)}(\lambda)) = \{0\}$ .*

PROOF. Let  $\lambda \in P_+(S^c)$ ,  $v \in V(\lambda)_\mu$  and  $w \in V(\lambda)_\nu$ . By Lemma 6.6.1 and the commutation relations in Section 6.3, we compute

$$\begin{aligned} L_{\sigma\lambda;\lambda}\tau((C_{v;v_\lambda}^\lambda)^\ast C_{w;v_\lambda}^\lambda) &= \tau(C_{v_\sigma\lambda;v_\lambda}^\lambda (C_{v;v_\lambda}^\lambda C_{v_\sigma\lambda;v_\lambda}^\lambda)^\ast C_{w;v_\lambda}^\lambda) \\ &= q^{(\lambda,\lambda)-(\mu,\sigma\lambda)}\tau(C_{v_\sigma\lambda;v_\lambda}^\lambda (C_{v;v_\lambda}^\lambda)^\ast (C_{v_\sigma\lambda;v_\lambda}^\lambda)^\ast C_{w;v_\lambda}^\lambda) \\ &= q^{2(\lambda,\lambda)-2(\mu,\sigma\lambda)}\tau((C_{v;v_\lambda}^\lambda)^\ast C_{v_\sigma\lambda;v_\lambda}^\lambda)\tau((C_{v_\sigma\lambda;v_\lambda}^\lambda)^\ast C_{w;v_\lambda}^\lambda), \end{aligned}$$

where Proposition 6.3.2(i) is used in the second equality and Proposition 6.3.2(ii) is used in the first and third equality. This computation, together with the injectivity of  $L_{\sigma\lambda;\lambda}$ , shows that it suffices to give a proof of the lemma for the special case that  $v = v_{\sigma\lambda}$ . So we fix  $h \in H_{\gamma_0(\lambda)}(\lambda)$  and  $w \in V(\lambda)_\nu$  with  $\nu \in P(\lambda)$  and  $\nu \neq \sigma\lambda$ . It follows from Lemma 6.6.3 that  $\tilde{h} := \tau((C_{v_\sigma\lambda;v_\lambda}^\lambda)^\ast C_{w;v_\lambda}^\lambda)h$  is an eigenvector of  $L_{\sigma\lambda;\lambda}$  with eigenvalue  $\tilde{\gamma}_0(\lambda) = q^{2(\lambda,\sigma^{-1}(\nu)-\lambda)}\gamma_0(\lambda)$ . By Proposition 6.5.3 we have  $\tilde{\gamma}_0(\lambda) > \gamma_0(\lambda)$ , hence  $\tilde{h} = 0$  by the maximality of the eigenvalue  $\gamma_0(\lambda)$ .  $\square$

COROLLARY 6.6.8.  $\gamma_0(\lambda) = 1$  for all  $\lambda \in P_+(S^c)$ .

PROOF. Follows from (6.3.10) and Lemma 6.6.7.  $\square$

The linear subspace of  $C_q[U]$  spanned by the matrix elements  $\{C_{\sigma\mu;\mu}^\mu\}_{\mu \in P_+}$  is a subalgebra of  $C_q[U]$  with algebraic generators  $C_{\sigma\varpi_i;\varpi_i}^{\varpi_i}$  ( $i \in [1, r]$ ). This follows from the fact that  $C_{\sigma\mu;\mu}^\mu C_{\sigma\nu;\nu}^\nu = \lambda_{\mu,\nu} C_{\sigma(\mu+\nu);\mu+\nu}^{\mu+\nu}$ , where the scalar  $\lambda_{\mu,\nu} \in \mathbb{T}$  depends on the particular choices of orthonormal bases for the finite dimensional irreducible representations  $V(\mu)$  and  $V(\nu)$  (cf. [110, Proof of Proposition 3.12]). Then it follows from Proposition 6.3.2 and Lemma 6.6.1 that

$$(6.6.6) \quad L_{\sigma(\mu+\nu);\mu+\nu} = L_{\sigma\mu;\mu} L_{\sigma\nu;\nu}$$

for all  $\mu, \nu \in P_+(S^c)$ , hence  $\text{span}\{L_{\sigma\lambda;\lambda}\}_{\lambda \in P_+(S^c)}$  is a commutative subalgebra of  $\mathcal{B}(H)$ . Set

$$(6.6.7) \quad H_1 := \bigcap_{i \in S^c} H_1(\varpi_i),$$

then  $H_1 \subset H_1(\lambda)$  for all  $\lambda \in P_+(S^c)$  by (6.6.6).

LEMMA 6.6.9.  $H_1 = H_1(\lambda)$  for all  $\lambda \in P_{++}(S^c)$ . In particular,  $H_1 \neq \{0\}$ .

PROOF. For  $\mu \in P_+(S^c)$  we have  $\|L_{\sigma\mu;\mu}\| = 1$ . Moreover, for any  $h \in H$ ,

$$(6.6.8) \quad h \in H_1(\mu) \Leftrightarrow \|L_{\sigma\mu;\mu}h\| = \|h\|.$$

This follows from the eigenspace decomposition (6.6.5) for  $L_{\sigma\mu;\mu}$  and the fact that 1 is the largest eigenvalue of  $L_{\sigma\mu;\mu}$ . Let  $\lambda \in P_{++}(S^c)$  and choose arbitrary  $i \in S^c$ . Then  $\lambda = \mu + \varpi_i$  for certain  $\mu \in P_+(S^c)$ . By (6.6.6), we obtain for  $h \in H_1(\lambda)$ ,

$$\begin{aligned} \|h\| &= \|L_{\sigma\lambda;\lambda}h\| = \|L_{\sigma\mu;\mu}L_{\sigma\varpi_i;\varpi_i}h\| \\ &\leq \|L_{\sigma\varpi_i;\varpi_i}h\| \leq \|h\|, \end{aligned}$$

hence we have equality everywhere. By (6.6.8), it follows that  $h \in H_1(\varpi_i)$ . Since  $i \in S^c$  was arbitrary, we conclude that  $h \in H_1$ .  $\square$

LEMMA 6.6.10. *Let  $\lambda \in P_+(S^c)$ . For all  $v \in V(\lambda)_\mu$  with  $\mu \neq \sigma\lambda$  we have*

$$\tau((C_{v;v\lambda}^\lambda)^* C_{v\sigma\lambda;v\lambda}^\lambda)(H_1) \subset H_1^\perp.$$

PROOF. Let  $\Lambda \in P_{++}(S^c)$ ,  $\lambda \in P_+(S^c)$ , and  $v \in V(\lambda)_\mu$  with  $\mu \neq \sigma\lambda$  and  $\mu \in P(\lambda)$ . Then

$$(6.6.9) \quad L_{\sigma\Lambda;\Lambda} \tau((C_{v;v\lambda}^\lambda)^* C_{v\sigma\lambda;v\lambda}^\lambda) = q^{2(\Lambda, \lambda - \sigma^{-1}(\mu))} \tau((C_{v;v\lambda}^\lambda)^* C_{v\sigma\lambda;v\lambda}^\lambda) L_{\sigma\Lambda;\Lambda}$$

by Lemma 6.6.3. By Lemma 6.5.4 we have  $(\Lambda, \lambda - \sigma^{-1}(\mu)) > 0$ . Hence,

$$\begin{aligned} \tau((C_{v;v\lambda}^\lambda)^* C_{v\sigma\lambda;v\lambda}^\lambda)(H_1) &= \tau((C_{v;v\lambda}^\lambda)^* C_{v\sigma\lambda;v\lambda}^\lambda)(H_1(\Lambda)) \\ &\subset \bigoplus_{\gamma < 1} H_\gamma(\Lambda) = H_1(\Lambda)^\perp = H_1^\perp, \end{aligned}$$

which completes the proof of the lemma.  $\square$

COROLLARY 6.6.11.  $\dim(H_1) = 1$ .

PROOF. By Lemma 6.6.7 and Lemma 6.6.10 we obtain for any  $0 \neq h \in H_1$ ,

$$\overline{\tau(A_S)h} \subset \text{span}\{h\} \oplus H_1^\perp,$$

where the overbar means closure. Since  $\tau$  is irreducible, it follows that  $\text{span}\{h\} = H_1$ .  $\square$

Any vector  $h \in H_1$  with  $\|h\| = 1$  can serve now as the analogue in the representation space  $H$  of the distinguished vector  $e_0^{\otimes l(\sigma)}$  in the representation space of  $\pi_\sigma$ . By comparing the Gelfand-Naimark-Segal states of  $\tau$  and  $\pi_\sigma$  taken with respect to the cyclic vector  $h \in H_1$  ( $\|h\| = 1$ ) respectively  $e_0^{\otimes l(\sigma)}$ , we obtain the following lemma.

LEMMA 6.6.12. *We have  $\tau \simeq \pi_\sigma$  as irreducible \*-representations of  $A_S$ .*

PROOF. Fix an  $h \in H_1$  with  $\|h\| = 1$ , and define the Gelfand-Naimark-Segal states  $\phi_\tau, \phi_{\pi_\sigma} : A_S \rightarrow \mathbb{C}$  by

$$(6.6.10) \quad \phi_\tau(a) := (\tau(a)h, h), \quad \phi_{\pi_\sigma}(a) := (\pi_\sigma(a)e_0^{\otimes l(\sigma)}, e_0^{\otimes l(\sigma)}).$$

Then we have for  $\phi = \phi_\tau$  (respectively  $\phi = \phi_{\pi_\sigma}$ ),

$$(6.6.11) \quad \phi((C_{\mu,i;\lambda}^\lambda)^* C_{\nu,j;\lambda}^\lambda) = \delta_{\mu,\sigma\lambda} \delta_{\nu,\sigma\lambda}$$

for  $\lambda \in P_+(S^c)$ ,  $\mu, \nu \in P(\lambda)$ ,  $i \in [1, \dim(V(\lambda)_\mu)]$ , and  $j \in [1, \dim(V(\lambda)_\nu)]$ . Indeed, (6.6.11) for  $\phi = \phi_\tau$  follows from Lemma 6.6.7 and Lemma 6.6.10. For  $\phi = \phi_{\pi_\sigma}$ , recall that  $\pi_\sigma$  is an irreducible \*-representation of  $A_S$  (Proposition 6.5.5). We have seen in the previous section that  $L_{\sigma\lambda;\lambda} = \pi_\sigma((C_{\sigma\lambda;\lambda}^\lambda)^* C_{\sigma\lambda;\lambda}^\lambda)$  is injective for all  $\lambda \in P_+(S^c)$ , hence  $\sigma\lambda \in D(\lambda)$  (cf. (6.6.2)) for all  $\lambda \in P_+(S^c)$ . By (6.5.4), we actually have  $\sigma\lambda \in D_m(\lambda)$  for all  $\lambda \in P_+(S^c)$ . Hence the labeling  $\sigma \in W^S$  of  $\pi_\sigma$  coincides with its (unique) minimal coset representative defined in Lemma 6.6.5. Furthermore, the one dimensional

subspace  $H_1$  for  $\pi_\sigma$  is equal to  $\text{span}\{e_0^{\otimes l(\sigma)}\}$  (cf. Proposition 6.5.2(ii), Lemma 6.6.11). So (6.6.11) for  $\phi = \phi_{\pi_\sigma}$  follows again from Lemma 6.6.7 and Lemma 6.6.10.

By linearity it follows from (6.6.11) that  $\phi_\tau = \phi_{\pi_\sigma}$ , hence  $\tau$  and  $\pi_\sigma$  are unitarily equivalent \*-representations (cf. [26, Proposition 2.4.1]).  $\square$

The results of this section may be summarized as follows.

**THEOREM 6.6.13.** *For all  $S \subsetneq \Delta$ ,  $\{\pi_\sigma\}_{\sigma \in W^S}$  is a complete set of mutually inequivalent, irreducible \*-representations of the factorized \*-subalgebra  $A_S$ .*

Combining Proposition 6.5.7, Theorem 6.4.10 and Theorem 6.6.13 we obtain the following theorem.

**THEOREM 6.6.14.**  *$\{\pi_\sigma\}_{\sigma \in W^S}$  is a complete set of mutually inequivalent, irreducible \*-representations of  $\mathbb{C}_q[U/K_S]$  in the following cases:*

- (i)  $S = \emptyset$ , i.e.  $U/K_S = U/T$  is the full flag manifold;
- (ii)  $\#S^c = 1$  and the simple root  $\alpha \in S^c$  is a Gelfand node.

*For these cases the restriction to  $\mathbb{C}_q[U/K_S]$  of the universal  $C^*$ -norm on  $\mathbb{C}_q[U]$  coincides with the universal  $C^*$ -norm on  $\mathbb{C}_q[U/K_S]$ .*

In fact, Theorem 6.6.14 is true for all generalized flag manifolds  $U/K_S$  in view of Remark 6.4.11.





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